Equilibrium states in negative curvature

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Abstract

With their origin in thermodynamics and symbolic dynamics, Gibbs measures are crucial tools to study the ergodic theory of the geodesic flow on negatively curved manifolds. We develop a framework (through Patterson-Sullivan densities) allowing us to get rid of compactness assumptions on the manifold, and prove many existence, uniqueness and finiteness results of Gibbs measures. We give many applications to the variational principle, the counting and equidistribution of orbit points and periods, the unique ergodicity of the strong unstable foliation and the classification of Gibbs densities on some Riemannian covers. [1]

1 Introduction

With their thermodynamic origin, Gibbs measures (or states) are very useful in symbolic dynamics over finite alphabets (see for instance [Rue3, Zin, Kel]). Sinai, Bowen and Ruelle introduced them in hyperbolic dynamics (which, via coding theory, has strong links to symbolic dynamics), in particular in order to study the weighted distribution of periodic orbits and the equilibrium states for given potentials (see for instance [Sin, Bow4, BoR, PaPo2]). For instance, this allows the dynamical analysis of the geodesic flow of negatively curved Riemannian manifolds, provided its nonwandering set is compact. As a first step to venture beyond the compact case, Gibbs measures in symbolic dynamics over countable alphabets have been developped by Sarig [Sar1, Sar3]. But since no coding theory which does not lose geometric information is known for general noncompact manifolds, this methodology is not well adapted to the noncompact case. Note that a coding-free approach of transfer operators, in particular via improved spectral methods, was put in place by Dolgopyat, Liverani, Baladi, Gouëzel and others (see for instance, restricting the references to the case of flows, [Dol, Live, BuL, Tsu1, Tsu2, BaLi, GLP], as well as [AvG] for an example in a non locally homogeneous and noncompact situation). In this text, we construct and study geometrically Gibbs measures for the geodesic flow of negatively curved Riemannian manifolds, without compactness assumption.

By considering the action on a universal Riemannian cover of its covering group, this study is strongly related to the (weighted) distribution of orbits of discrete groups of isometries of negatively curved simply connected Riemannian manifolds. After work of Huber and Selberg (in particular through his trace formula) in constant curvature, Margulis (see for instance [Marg]) made a breakthrough, albeit in the unweighted lattice case, to

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give the precise asymptotic growth of the orbits. Patterson [Patt] and Sullivan [Sul1], in the surface and constant curvature case respectively, have introduced a nice approach using measures at infinity, giving in particular a nice construction of the measure of maximal entropy in the cocompact case. The Patterson-Sullivan theory extends to variable curvature and non lattice case, and an optimal reference (in the unweighted case) is due to Roblin [Rob1].

After preliminary work of Hamenstädt [Ham2] on Gibbs cocycles, of Ledrappier [Led2], Coudene [Cou2] and Schapira [Sch3] using a slightly different approach, and of Mohsen [Moh], the aim of this paper is to develop an optimal theory of Gibbs measures for the dynamical study of geodesic flows in general negatively curved Riemannian manifolds, and of weighted distribution of orbits of general discrete groups of isometries of negatively curved simply connected Riemannian manifolds.

Let us give a glimpse of our results, starting by describing the players. Let M be a complete connected Riemannian manifold with pinched negative curvature. Let $F : T^1M \to \mathbb{R}$ be a Hölder-continuous map, called a *potential*, which is going to help us define the various weights. In order to simplify the exposition of this introduction, we assume here that F is bounded and invariant by the antipodal map $v \mapsto -v$ (see the body of the text for complete statements). Let $p : \widetilde{M} \to M$ be a universal Riemannian covering map, with covering group Γ and sphere at infinity $\partial_\infty \widetilde{M}$, and let $\widetilde{F} = F \circ p$. We make no compactness assumption on M: we only assume Γ to be non virtually nilpotent. (In the body of the text, we will also allow M to be a good orbifold, hence Γ to have torsion.) Let $\phi = (\phi_t)_{t\in\mathbb{R}}$ be the geodesic flow on T^1M and $\widetilde{\phi} = (\widetilde{\phi}_t)_{t\in\mathbb{R}}$ the one on $T^1\widetilde{M}$. For all $x, y \in \widetilde{M}$, define $\int_x^y \widetilde{F} = \int_0^{d(x,y)} \widetilde{F}(\widetilde{\phi}_t v)\; dt$ where v is the unit tangent vector at x to a geodesic from x through y. For every periodic orbit g of ϕ, let $\mathscr{L}_g$ be the Lebesgue measure along g and $\int_g F = \mathscr{L}_g(F)$ the *period* of g for the potential F.

We will study three numerical invariants of the weighted dynamics.

• Let $x, y \in \widetilde{M}$. The *critical exponent* of (Γ, F) is an exponential growth rate of the orbit points of Γ weighted by the potential F:

$$\delta_{\Gamma,\,F} = \lim_{t\to+\infty} \frac{1}{t} \log \sum_{\gamma\in\Gamma,\; d(x,\gamma y)\leq t} e^{\int_x^{\gamma y} \widetilde{F}}\;.$$

• For every $t \geq 0$, let $\mathscr{P}\mathrm{er}(t)$ be the set of periodic orbits of ϕ with length at most t. Let W be a relatively compact open subset of T^1M meeting the nonwandering set of ϕ. The *Gurevich pressure* of (M, F) is an exponential growth rate of the closed geodesics of M weighted by the potential F:

$$P_{Gur}(M, F) = \lim_{t\to+\infty} \frac{1}{t} \log \sum_{g\in\mathscr{P}\mathrm{er}(t),\; g\cap W\neq\emptyset} e^{\int_g F}\;.$$

• Let $\mathscr{M}$ be the set of ϕ-invariant probability measures on T^1M and let $h_m(\phi)$ be the (metric) entropy of the geodesic flow ϕ with respect to $m \in \mathscr{M}$. The *topological pressure* of (ϕ, F) is the upper bound of the sum of the entropy and the averaged potential of all states:

$$P_{top}(\phi, F) = \sup_{m\in\mathscr{M}} \Big(h_m(\phi) + \int_{T^1M} F\, dm\Big)\;.$$

When $F = 0$, the quantity $\delta_{\Gamma,\,F}$ is the usual critical exponent δ_Γ of Γ, and $P_{top}(\phi, F)$ is the topological entropy $h_{top}(\phi)$ of ϕ. We prove in Subsection 4.2 and 4.3 that the above limits do exist and are independent of x, y, W, a result of Roblin [Rob2] when $F = 0$. We have not even assumed Γ to be finitely generated, so we prove in Subsection 4.4 that $\delta_{\Gamma,\,F}$ is the upper bound of the critical exponents with potential F of the free finitely generated semigroups in Γ (of Schottky type), a result due to Mercat [Mer] if $F = 0$. For Markov shifts on countable alphabets, the Gurevich pressure has been introduced by Gurevich [Gur1, Gur2] when the potential is equal to 0, and by Sarig [Sar1, Sar3, Sar2] in general. The Gurevich pressure had not been studied much in a non symbolic noncompact context before this work.

Our first result, generalising in particular results of Manning [Mann] ($\delta_\Gamma = h_{top}(\phi)$), Ruelle [Rue2] and Parry when M is compact, is the following one (see Subsection 4.3 and Chapter 6).

Theorem 1.1 *We have*

$$P_{Gur}(M, F) = \delta_{\Gamma,\,F} = P_{top}(\phi, F)\ .$$

Let us now define the Gibbs measures (see Chapter 3). We use Mohsen's extension (whose compactness assumptions are not necessary) of the Patterson-Sullivan construction, see [Moh], which is more convenient than the one of [Led2, Cou2, Sch3]. The (normalised) *Gibbs cocycle* of (M, F) is the map C from $\partial_\infty\widetilde{M} \times \widetilde{M} \times \widetilde{M}$ to $\mathbb{R}$ defined by

$$(\xi, x, y) \mapsto C_\xi(x, y) = \lim_{t\to+\infty} \int_y^{\xi_t} (\widetilde{F} - \delta_{\Gamma,\,F}) - \int_x^{\xi_t} (\widetilde{F} - \delta_{\Gamma,\,F})\ ,$$

where $t \mapsto \xi_t$ is any geodesic ray converging to a given $\xi \in \partial_\infty\widetilde{M}$. When $F = 0$, we have $C_\xi(x, y) = \delta_\Gamma \beta_\xi(x, y)$ where β is the Busemann cocycle. A (normalised) *Patterson density* of (Γ, F) is a family of pairwise absolutely continuous finite positive nonzero Borel measures $(\mu_x)_{x\in\widetilde{M}}$ on the sphere at infinity $\partial_\infty\widetilde{M}$ such that, for all $\gamma \in \Gamma$, $x, y \in \widetilde{M}$ and $\xi \in \partial_\infty\widetilde{M}$,

$$\gamma_*\mu_x = \mu_{\gamma x} \quad\text{and}\quad \frac{d\mu_x}{d\mu_y}(\xi) = e^{-C_\xi(x,y)}\ .$$

Mohsen has extended Patterson's construction of such a family, and Sullivan's shadow lemma. It would be interesting to know when these Patterson densities associated to potentials are harmonic measures for random walks on an orbit of Γ, see [CM] when M is compact. It would also be interesting to extend to the case with nonzero potential the study of the spectral theory of Patterson densities, as done by Patterson and Sullivan in constant curvature.

Fix $x_0 \in \widetilde{M}$. For every $v \in T^1\widetilde{M}$, let v_-, v_0, v_+ be respectively the point at $-\infty$, the origin, the point at $+\infty$ of the geodesic line defined by v, and let t be the distance between v_0 and the closest point to x_0 on this line. Define a measure $\widetilde{m}_F$ on $T^1\widetilde{M}$ by

$$d\widetilde{m}_F(v) = \frac{d\mu_{x_0}(v_-)}{e^{C_{v_-}(v_0,x_0)}}\,\frac{d\mu_{x_0}(v_+)}{e^{C_{v_+}(v_0,x_0)}}\,dt\ .$$

This measure on $T^1\widetilde{M}$ is invariant under Γ and $\widetilde{\phi}$, hence induces a measure on T^1M invariant under ϕ, called the *Gibbs measure* (or *Gibbs state*) of the potential F. In symbolic

dynamics, the Gibbs measures are shift-invariant measures, which give to any cylinder a weight defined by the Birkhoff sum of the potential. Here, the analogs of the cylinders are the neighbourhoods of a (pointed) geodesic line, defined by small neighbourhoods of its two points at infinity, that we also weight according to the Birkhoff integral of the potential in order to define our Gibbs measures. In Subsection 3.8, we prove that our Gibbs measures indeed satisfy the Gibbs property on the dynamical balls.

We have a huge choice of potential functions. For instance, define

$$F^{su} = -\frac{d}{dt}_{|t=0} \log \operatorname{Jac}\big(\phi_{t|W^{su}(v)}\big)(v) \;,$$

the opposite of the pointwise exponential growth rate of the jacobian of the geodesic flow restricted to the strong unstable manifold (see Chapter 7). When M has dimension n and constant curvature -1, then F^{su} is constant, equal to $-(n-1)$. Some references use the opposite sign convention on the potential to define the topological pressure as $\sup_{m\in\mathscr{M}}\big(h_m(\phi) - \int_{T^1M} F\, dm\big)$, hence the unstable jacobian needs also to be defined with the opposite sign as the one above.

When M is compact, we recover, up to a scalar multiple, for $F = 0$ (the Patterson-Sullivan construction of) the Bowen-Margulis measure, and for $F = F^{su}$ the Liouville measure. Using Gibbs measures is a way of interpolating between these measures, and gives a wide range of applications. It would be interesting to study when our Gibbs measures are harmonic measures associated to elliptic second-order differential operators on M (see for instance [Ham3] when M is compact).

We will prove (see Chapter 7) new results about when the Liouville measure is a Gibbs measure. We refer for instance [PoY, Chap. 16] and [AaN] (see also Chapter 7) for the definition of the multiple recurrence property introduced by Furstenberg in his proof of Szemeredi's theorem, which is a reinforcement of the complete conservativity property. Note that there are many examples of noncompact (with infinite volume) Riemannian covers of compact Riemmanian manifolds with constant sectional curvature -1 and with ergodic (hence completely conservative) geodesic flow for the Liouville measure, see for instance [Ree]. It would be interesting to know when exactly the geodesic flow of a (non necessarily regular) Riemannian cover of a compact connected negatively curved Riemannian manifold is ergodic or 2-recurrent with respect to the Liouville measure, or to a given Gibbs measure.

Theorem 1.2 *If M is a Riemannian cover of a compact manifold, and if the geodesic flow ϕ of M is 2-recurrent with respect to the Liouville measure, then the Liouville measure is, up to a scalar multiple, the Gibbs measure for the potential F^{su} and $P_{Gur}(M, F^{su}) = \delta_{\Gamma,\,F^{su}} = P_{top}(\phi, F^{su}) = 0$.*

Let us now state (a simplified version of) the main results of our paper. An *equilibrium state* for the potential F is a ϕ-invariant probability measure m on T^1M such that $h_m(\phi) + \int_{T^1M} F\, dm$ is equal to the topological pressure $P_{top}(\phi, F)$ (hence realizing the upper bound defining it). The first result (see Chapter 6) says that the equilibrium states are the Gibbs states (normalised to probability measures).

Theorem 1.3 (Variational principle) *If m_F is finite, then $\frac{m_F}{\|m_F\|}$ is the unique equilibrium state. Otherwise, there exists no equilibrium state.*

When Γ is convex-cocompact, this result is due to Bowen and Ruelle [BoR]. Their approach, using symbolic dynamics, called the *thermodynamic formalism*, is described in many references, and has produced many extensions for various situations (see for instance [Rue3, Zin, Kel]), under some compactness assumption (except for the work of Sarig already mentioned). For general Γ but when $F = 0$, the result is due to Otal and Peigné [OtP], and we will follow their proof.

Up to translating F by a constant, which does not change the Patterson density nor the Gibbs measure, we assume, from now on in this introduction, that $\delta_{\Gamma,\,F} > 0$. The next result, of equidistribution of (ordered) pairs of Dirac masses on an orbit weighted by the potential towards the product of Patterson measures, is due to Roblin [Rob1] when $F = 0$, and we will follow his proof (as well as for the next three results). We denote by $\mathscr{D}_z$ the unit Dirac mass at a point $z \in \widetilde{M}$.

Theorem 1.4 (Two point orbital equidistribution) *If m_F is finite and mixing, then we have, as t goes to $+\infty$,*

$$\delta_{\Gamma,\,F}\,\|m_F\|\;e^{-\delta_{\Gamma,\,F}\,t}\sum_{\gamma\in\Gamma\,:\,d(x,\gamma y)\le t} e^{\int_x^{\gamma y}\widetilde F}\;\mathscr{D}_{\gamma y}\otimes\mathscr{D}_{\gamma^{-1}x}\;\overset{*}{\rightharpoonup}\;\mu_x\otimes\mu_y\;.$$

Corollary 1.5 (Orbital counting) *If m_F is finite and mixing, then, as t goes to $+\infty$,*

$$\sum_{\gamma\in\Gamma\,:\,d(x,\gamma y)\le t} e^{\int_x^{\gamma y}\widetilde F}\;\sim\;\frac{\|\mu_y\|\,\|\mu_x\|}{\delta_{\Gamma,\,F}\,\|m_F\|}\;e^{t\,\delta_{\Gamma,\,F}}\;.$$

When M is compact and $F = 0$, this corollary is due to Margulis. When $F = 0$, it is due to Roblin in this generality. Theorem 1.4 also gives counting asymptotics of orbit points γx_0 staying, as well as their reciprocal $\gamma^{-1}x_0$, in given cones, see Subsection 9.2 (improving, under stronger hypotheses, the logarithmic growth result of Subsection 4.2).

The next two results are due to Bowen [Bow2] when M is compact (or even Γ convex-cocompact), see also [PaPo1, Par2]. But our proof is very different even under this compactness assumption, in particular we avoid any coding theory, hence any symbolic dynamics, and any zeta function or transfer operator. They are due to Roblin [Rob1] when $F = 0$ under this generality.

Theorem 1.6 (Equidistribution of periodic orbits) *If m_F is finite and mixing, then, as t goes to $+\infty$,*

$$\delta_{\Gamma,\,F}\;e^{-\delta_{\Gamma,\,F}\,t}\sum_{g\in\mathscr{P}\mathrm{er}(t)} e^{\int_g F}\;\mathscr{L}_g\;\overset{*}{\rightharpoonup}\;\frac{m_F}{\|m_F\|}\;.$$

Corollary 1.7 (Counting of periodic orbits) *If Γ is geometrically finite and if m_F is finite and mixing, then, as t goes to $+\infty$,*

$$\sum_{g\in\mathscr{P}\mathrm{er}(t)} e^{\int_g F}\;\sim\;\frac{e^{t\,\delta_{\Gamma,\,F}}}{t\;\delta_{\Gamma,\,F}}\;.$$

Let us now describe additional results. Consider the Poincaré series of (Γ, F)

$$Q_{\Gamma,\,F,\,x,\,y}(s) = \sum_{\gamma\in\Gamma} e^{\int_x^{\gamma y}(\widetilde F - s)}\;,$$

which, independently of $x, y \in \widetilde{M}$, converges if $s > \delta_{\Gamma,F}$ and diverges if $s < \delta_{\Gamma,F}$. In Chapter 5, extending works of Hopf, Tsuji, Sullivan, Roblin when $F = 0$ and following the proofs of [Rob1], we give criteria for the ergodicity and non ergodicity of the geodesic flow on T^1M endowed with a Gibbs measure. It would be interesting to know when the geodesic flow on T^1M is multiply recurrent with respect to the Gibbs measure associated to a given potential.

Theorem 1.8 *The following conditions are equivalent*

(i) The Poincaré series of (Γ, F) *diverges at* $s = \delta_{\Gamma,F}$.

(ii) The conical limit set of Γ *has positive measure with respect to* μ_x.

(iii) The dynamical system $(\partial_\infty\widetilde{M} \times \partial_\infty\widetilde{M}, \Gamma, \mu_x \otimes \mu_x)$ *is ergodic and completely conservative.*

(iv) The dynamical system (T^1M, ϕ, m_F) *is ergodic and completely conservative.*

We discuss several applications in Subsection 5.3. In particular, if the Poincaré series diverges at $s = \delta_{\Gamma,F}$, then the normalised Patterson density has no atom and is unique up to a scalar multiple, and the Gibbs measure on T^1M is unique up to a scalar multiple and ergodic when finite.

In view of (iii), it would be interesting to study if $(\partial_\infty\widetilde{M}, \mu_x)$ is a (weak) Γ-boundary in the sense of Burger-Mozes [BuM] or Bader-Furman [BaF, BaFS].

In Subsection 8.2, we give a criterion for the finiteness of m_F when M is geometrically finite, extending the one in [DaOP] when $F = 0$, as in Coudene [Cou2]. The results 1.4 to 1.7 require m_F to be finite and mixing. But given the finiteness assumption, the mixing one is mild. Indeed, by Babillot's result [Bab2, Theo. 1], since Gibbs measures are quasi-product measures, if m_F is finite, then m_F is mixing if and only if the geodesic flow ϕ is topologically mixing on T^1M, a purely topological condition which is seemingly easier to check and is conjecturally always satisfied.

It is indeed another interesting feature of Gibbs measures, that they have strong ergodic properties. As soon as they are finite, they are ergodic and even mixing as just said (when ϕ is topologically mixing on T^1M). If in addition $P_{top}(\phi, F) > \sup F$ (for example if $\sup F - \inf F < h_{top}(\phi)$), then the entropy of the Gibbs measure m_F is positive. These properties are natural, since the geodesic flow in negative curvature is a hyperbolic flow, and therefore should have strong stochastic properties. However, generically (in Baire's sense), Coudene-Schapira [CS] have announced that the invariant probability measures for ϕ are ergodic, but not mixing and of entropy zero. Therefore, contrarily to the generic case, the Gibbs measures provide an explicit family of good measures reflecting the strong stochastic properties of the geodesic flow.

Consider the cocycle c_F defined on the (ordered) pairs of vectors $v, w \in T^1M$ in the same leaf of the strong unstable foliation $\mathscr{W}^{su}$ of the geodesic flow ϕ in T^1M by

$$c_F : (w, v) \mapsto \lim_{t \to +\infty} \int_0^t \big(F(\phi_{-s}w) - F(\phi_{-s}v)\big)\, ds\ .$$

A nonzero family $(\nu_T)_T$ of locally finite (positive Borel) measures on the transversals T to $\mathscr{W}^{su}$, stable by restrictions, is c_F*-quasi-invariant* if for every holonomy map $f : T \to T'$

along the leaves of $\mathscr{W}^{su}$ between two transversals T, T' to $\mathscr{W}^{su}$, we have

$$\frac{d\, f_*\nu_T}{d\,\nu_{T'}}(f(x)) = e^{c_F(f(x),\,x)}\ .$$

We will prove in Chapter 10, as in [Rob1, Chap. 6] when $F = 0$ and in [Sch1, Chap. 8.2.2] (unpublished), the following unique ergodicity result.

Theorem 1.9 *If m_F is finite and mixing, then there exists, up to a scalar multiple, one and only one c_F-quasi-invariant family of transverse measures $(\mu_T)_T$ for $\mathscr{W}^{su}$ such that μ_T gives full measure to the set of elements of T which are negatively recurrent under the geodesic flow ϕ in T^1M.*

We give an explicit construction of $(\mu_T)_T$ in terms of the above Patterson densities, and we deduce from Theorem 1.9, when M is geometrically finite, the classification of the ergodic c_F-quasi-invariant families of transverse measures. This generalises works when $F = 0$ of Furstenberg, Dani, Dani-Smillie, Ratner for M a surface, and of Roblin for general M.

The main tool, which follows from Babillot's work, is the following equidistribution result in T^1M of the strong unstable leaves towards the Gibbs measure m_F: for every $v \in T^1M$, let $W^{su}(v)$ be its strong unstable leaf for ϕ and let $(\mu_{W^{su}(v)})_{v\in T^1M}$ be the family of conditional measures on the strong unstable leaves of the Gibbs measure m_F associated to the potential F; then for every relatively compact Borel subset K of $W^{su}(v)$ (containing a positively recurrent vector in its interior), for every $\psi \in \mathscr{C}_c(T^1M;\mathbb{R})$, we have

$$\lim_{t\to+\infty} \frac{1}{\int_K e^{c_F(w,v)}\, d\mu_{W^{su}(v)}(w)} \int_K \psi\circ\phi_t(w)\, e^{c_F(w,v)}\, d\mu_{W^{ss}(v)}(w) = \frac{1}{\|m_F\|}\int_{T^1M} \psi\, dm_F\ .$$

We also prove and use the subexponential growth of the mass of strong unstable balls for the conditional measures on the strong unstable leaves (we may no longer have polynomial growth as when $F = 0$).

In the final chapter 11, we extend the work of Babillot-Ledrappier [BaL], Hamenstädt [Ham4] and Roblin [Rob3] when $F = 0$ to study the Gibbs states on Galois Riemannian covers of M.

Let $\chi : \Gamma \to \mathbb{R}$ be a (real) character of Γ. For all $x, y \in \widetilde{M}$, define the *(twisted) Poincaré series of* (Γ, F, χ) as

$$Q_{\Gamma,\,F,\,\chi,\,x,\,y}(s) = \sum_{\gamma\in\Gamma} e^{\chi(\gamma)+\int_x^{\gamma y}(\widetilde{F}-s)}\ .$$

The *(twisted) critical exponent of* (Γ, F, χ) is the element $\delta_{\Gamma,\,F,\,\chi}$ in $[-\infty, +\infty]$ defined by

$$\delta_{\Gamma,\,F,\,\chi} = \limsup_{n\to+\infty} \frac{1}{n}\log \sum_{\gamma\in\Gamma,\; n-1<d(x,\gamma y)\le n} e^{\chi(\gamma)+\int_x^{\gamma y}\widetilde{F}}\ .$$

When $F = 0$ and M is compact, since $H^1(M;\mathbb{R}) \simeq \mathrm{Hom}(\Gamma,\mathbb{R})$, we recover the Poincaré series associated with a cohomology class of M introduced in [Bab1], and the critical exponent of (Γ, F, χ) is then called the *cohomological pressure*.

A *(twisted) Patterson density* of dimension $\sigma \in \mathbb{R}$ for (Γ, F, χ) is a family of finite nonzero (positive Borel) measures $(\mu_x)_{x\in\widetilde{M}}$ on $\partial_\infty \widetilde{M}$ such that, for every $\gamma \in \Gamma$, for all $x, y \in \widetilde{M}$, for every $\xi \in \partial_\infty \widetilde{M}$, we have

$$\gamma_* \mu_x = e^{-\chi(\gamma)} \mu_{\gamma x} ,$$

$$\frac{d\mu_x}{d\mu_y}(\xi) = e^{\lim_{t\to+\infty} \int_x^{\xi_t}(\widetilde{F}-\sigma) - \int_y^{\xi_t}(\widetilde{F}-\sigma)} ,$$

where $t \mapsto \xi_t$ is any geodesic ray converging to ξ.

Using the above-mentioned extension of the Hopf-Tsuji-Sullivan-Roblin theorem (Theorem 1.8), as well as (an immediate extension to the case with potential of) the Fatou-Roblin radial convergence theorem for the ratio of the total masses of two Patterson densities (see [Rob3, Théo. 1.2.2]), we prove in Chapter 11, amongst other results, that the topological pressure is unchanged by taking an amenable cover of M, and we give a classification result of the Patterson densities on nilpotent covers of M, for instance when M is compact.

Theorem 1.10 *Let Γ' be a normal subgroup of Γ, and $F' : \Gamma'\backslash T^1\widetilde{M} \to \mathbb{R}$ be the map induced by $\widetilde{F}$.*

(1) If the quotient group Γ/Γ' is amenable, then $\delta_{\Gamma,F} = \delta_{\Gamma',F'}$.

(2) Let $\chi : \Gamma \to \mathbb{R}$ be a character of Γ. If $\delta_{\Gamma,F,\chi} < +\infty$ and $Q_{\Gamma,F,\chi,x,y}(s)$ diverges at $s = \delta_{\Gamma,F,\chi}$ (for instance if Γ is convex-cocompact), then there exists a unique (up to a scalar multiple) twisted Patterson density $\mu_{\Gamma,F,\chi} = (\mu_{\Gamma,F,\chi,x})_{x\in\widetilde{M}}$ of dimension $\delta_{\Gamma,F,\chi}$ for (Γ, F, χ).

(3) If Γ is convex-cocompact and Γ/Γ' is nilpotent, then the set of ergodic Patterson densities for (Γ', F') is the set of multiples of $\mu_{\Gamma,F,\chi}$ for the characters χ of Γ vanishing on Γ'.

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Contents

2 Background on negatively curved manifolds

In this text, the triple $(\widetilde{M}, \Gamma, \widetilde{F})$ will denote the following data.

Let $\widetilde{M}$ be a complete simply connected Riemannian manifold, with dimension at least 2 and pinched sectional curvature $-b^2 \leq K \leq -1$, where $b \geq 1$. Let Γ be a nonelementary (see the definition of nonelementary below) discrete group of isometries of $\widetilde{M}$. Let $\widetilde{F} : T^1\widetilde{M} \to \mathbb{R}$ be a Hölder-continuous Γ-invariant map, called a *potential* (see the definition of the Hölder-continuity we will use in Subsection 2.1).

We refer for instance to [BaGS, BrH] for general background on negatively curved manifolds. We only recall in this chapter the notation and results about them that we will use in this text.

General notation. Here is some general notation that will be used in this text.

Let A be a subset of a set E. We denote by $\mathbb{1}_A : E \to \{0, 1\}$ the characteristic function of A: $\mathbb{1}_A(x) = 1$ if $x \in A$, and $\mathbb{1}_A(x) = 0$ otherwise. We denote by ${}^cA = E - A$ the complementary subset of A in E.

We denote by $\log$ the natural logarithm (with $\log(e) = 1$).

For every real number x, we denote by $\lfloor x \rfloor$ the integral part of x, that is the largest integer not greater that x.

For every smooth map $f : N \to N'$ between smooth manifolds, we denote by $Tf : TN \to TN'$ its tangent map.

We denote by $\|\mu\|$ the total mass of a finite positive measure μ.

If $(X, \mathscr{A})$ and $(Y, \mathscr{B})$ are measurable spaces, $f : X \to Y$ a measurable map, and μ is a measure on X, we denote by $f_*\mu$ the image measure of μ by f, with $f_*\mu(A) = \mu(f^{-1}(A))$ for every $A \in \mathscr{A}$.

For every integer $n \geq 2$, we denote by $\mathbb{H}^n_{\mathbb{R}}$ (any model of) the real hyperbolic space of dimension n (its sectional curvature is constant with value -1).

Given a topological space X, we denote by $\mathscr{C}_c(X; \mathbb{R})$ the vector space of continuous maps $f : X \to \mathbb{R}$ with compact support.

2.1 Uniform local Hölder-continuity

Recall that two distances d and d' on a set E are (uniformly locally) *Hölder-equivalent* if there exist $c, \epsilon > 0$ and $\alpha \in \,]0, 1]$ such that for all $x, y \in E$ with $d(x, y) \leq 1$, we have

$$\frac{1}{c}\, d(x, y)^{\frac{1}{\alpha}} \leq d'(x, y) \leq c\, d(x, y)^{\alpha} \,.$$

This relation is an equivalence relation on the set of distances on E, and a *Hölder structure* on E is the choice of such an equivalence class. A metric space will be endowed with the Hölder structure of its distance. A map $f : E \to E'$ between sets endowed with Hölder structures is *Hölder-continuous* if for any distances d and d' in the Hölder structures of E and E', there exist $c, \epsilon > 0$ and $\alpha \in \,]0, 1]$ such that for all $x, y \in E$ with $d(x, y) \leq \epsilon$, we have

$$d'(f(x), f(y)) \leq c\, d(x, y)^{\alpha} \,.$$

We will call c, ϵ and α the *Hölder constants* of f, even though they are not unique and depend on the choices of the distances d and d' in their equivalence class.

A Hölder structure on a set E is *natural* under a group of bijections G of E if any element of G is Hölder-continuous.

Remarks. Note that this condition is a global one, and that some other texts only require the above inequality to hold, for some given c, α and for every x, only if y is close enough to x (that is, in some ball of center x whose radius might be smaller than 1 and might depend on x). When the topology on E defined by d is compact, these two definitions coincide, but we are trying to avoid any compactness assumption in this text. For instance in symbolic dynamics, the potentials may be only required to have this second property (or even less, see for instance [Kel]): since two sequences in the same strong stable leaf are eventually equal, the Gibbs cocycle is then well-defined. But the fact that the Gibbs cocycle in our case is well defined will use this global Hölder-continuity (see Subsection 3.3).

Let $f : X \to Y$ be a map between metric spaces. If X is geodesic, then the definition of the Hölder-continuity of f may be strengthened as follows: f is *Hölder-continuous* if and only if there exists $c > 0$ and $\alpha \in]0, 1]$ such that for all $x, y \in E$ with $d(x, y) \leq 1$, we have

$$d(f(x), f(y)) \leq c\, d(x, y)^\alpha \,.$$

We will use this definition throughout this text.

Another consequence of the global Hölder-continuous property of f is that f then has at most linear growth: if X is a geodesic space, the above condition implies that for all x, y in X with $d(x, y) \geq 1$, we have $d(f(x), f(y)) \leq 3c\, d(x, y)$. This will turn out to be useful in particular in Subsection 10.3.

2.2 Boundary at infinity, isometries and the Busemann cocycle

We denote by d or $d_{\widetilde M}$ the Riemannian distance on $\widetilde M$. We denote by $\partial_\infty \widetilde M$ the boundary at infinity of $\widetilde M$. We endow $\widetilde M \cup \partial_\infty \widetilde M$ with the cone topology, homeomorphic to the closed unit ball of $\mathbb{R}^n$, where n is the dimension of $\widetilde M$. For every $x \in \widetilde M$, recall that the Gromov-Bourdon *visual distance* d_x on $\partial_\infty \widetilde M$ seen from x (see [Bou]) is defined by

$$d_x(\xi, \eta) = \lim_{t \to +\infty} e^{\frac{1}{2}(d(x,y) - d(x, \xi_t) - d(x, \eta_t))} \,, \tag{1}$$

where $t \mapsto \xi_t, \eta_t$ are any geodesic rays ending at ξ, η respectively. By the triangle inequality, for all $x, y \in \widetilde M$ and $\xi, \eta \in \partial_\infty \widetilde M$, we have

$$e^{-d(x,y)} \leq \frac{d_x(\xi, \eta)}{d_y(\xi, \eta)} \leq e^{d(x,y)} \,. \tag{2}$$

We endow $\partial_\infty \widetilde M$ with the Hölder structure which is the equivalence class of any visual distance. It is natural under the isometry group of $\widetilde M$ and its induced topology on $\partial_\infty \widetilde M$ coincides with the cone topology.

For the following definitions, let $r > 0$, $x \in \widetilde M \cup \partial_\infty \widetilde M$, $A \subset \widetilde M$ and $B \subset \partial_\infty \widetilde M$. We denote by $\mathscr{O}_x A$ the *shadow of A seen from x*, that is, the subset of $\partial_\infty \widetilde M$ consisting of the endpoints of the geodesic rays (if $x \in \widetilde M$) or lines (if $x \in \partial_\infty \widetilde M$) starting from x and meeting A. Note that $\mathscr{O}$ stands for "ombre". We denote by $\mathscr{C}_x B$ the *cone on B with vertex x* in $\widetilde M$, that is, the union of the geodesic rays or lines starting from x and ending at B. We denote by $\mathscr{N}_r A$ the open r-neighbourhood of A, that is,

$$\mathscr{N}_r A = \{x \in \widetilde M \;:\; d(x, A) < r\} \,.$$

We denote by $\mathscr{N}_r^{\,-} A$ the *open r-interior* of A, that is,

$$\mathscr{N}_r^{\,-} A = \{x \in \widetilde{M} \;:\; d(x, {}^{c}A) > r\}\;,$$

which is contained in A.

For every isometry γ of $\widetilde{M}$, we denote by

$$\ell(\gamma) = \inf_{x\in\widetilde{M}} d(x, \gamma x)$$

the *translation length* of γ on $\widetilde{M}$. Recall that γ is *elliptic* if it has a fixed point in $\widetilde{M}$, *parabolic* if non elliptic and $\ell(\gamma) = 0$, and *hyperbolic* otherwise. In this last case, we denote by

$$Axe_\gamma = \{x \in \widetilde{M} \;:\; d(x, \gamma x) = \ell(\gamma)\}$$

the *translation axis* of γ, which is isometric to $\mathbb{R}$.

The isometry group $\operatorname{Isom}(\widetilde{M})$ of $\widetilde{M}$ is endowed with the compact-open topology. Let Γ' be a discrete group of isometries of $\widetilde{M}$. Its *limit set*, which is the set of accumulation points in $\partial_\infty\widetilde{M}$ of any orbit of Γ' in $\widetilde{M}$, will be denoted by $\Lambda\Gamma'$, and the convex hull in $\widetilde{M}$ of this limit set by $\mathscr{C}\Lambda\Gamma'$. Recall that Γ' is *nonelementary* if $\operatorname{Card}(\Lambda\Gamma') \geq 3$. Since $\widetilde{M}$ has pinched negative curvature, the following assertions are equivalent (see for instance [Bowd]):

- Γ' is nonelementary,
- Γ' contains a free subgroup of rank 2,
- Γ' is not virtually solvable,
- Γ' is not virtually nilpotent.

Also recall that Γ' is *convex-cocompact* if Γ' is nonelementary and $\Gamma'\backslash\mathscr{C}\Lambda\Gamma'$ is compact.

If Γ' is nonelementary, then $\Lambda\Gamma'$ is the closure of the set of fixed points of hyperbolic elements of Γ', and it is the smallest nonempty, closed, Γ'-invariant subset of $\partial_\infty\widetilde{M}$. In particular, if Γ'' is an infinite normal subgroup of Γ', then

$$\Lambda\Gamma'' = \Lambda\Gamma'\;.$$

The *conical* (or *radial*) *limit set* $\Lambda_c\Gamma'$ of Γ' is the set of points $\xi \in \partial_\infty\widetilde{M}$ such that there exists a sequence of orbit points of x under Γ' converging to ξ while staying at bounded distance from a geodesic ray ending at ξ. We have $\Lambda_c\Gamma' = \Lambda\Gamma'$ if Γ' is convex-cocompact. If Γ' is nonelementary, the conical limit set $\Lambda_c\Gamma'$ of Γ' is dense in its limit set $\Lambda\Gamma'$, and the point at infinity of a geodesic ray ρ in $\widetilde{M}$ belongs to $\Lambda_c\Gamma'$ if and only if the image of ρ in M comes back in some compact subset at times tending to $+\infty$.

A point ξ in $\partial_\infty\widetilde{M}$ is a *Myrberg point* of Γ' (see for instance [Tuk]) if for all $\eta \neq \eta'$ in $\partial_\infty\widetilde{M}$ and $x \in \widetilde{M}$, there exists a sequence $(\gamma_n)_{n\in\mathbb{N}}$ in Γ' such that $\lim_{n\to\infty}\gamma_n x = \eta$ and $\lim_{n\to\infty}\gamma_n\xi = \eta'$. We will denote by $\Lambda_{\rm Myr}\Gamma'$ the set of Myrberg points of Γ'. It is clearly a subset of $\Lambda_c\Gamma'$, and even a proper subset of $\Lambda_c\Gamma'$, since the fixed points of the hyperbolic elements of Γ' are conical limit points but not Myrberg points.

In this text, we are not assuming Γ to be torsion free.

Note that the action of Γ on $\Lambda\Gamma$ is not necessarily faithful (viewing the real hyperbolic plane $\mathbb{H}^2_\mathbb{R}$ as the intersection with the horizontal plane of the ball model of the real hyperbolic space $\mathbb{H}^3_\mathbb{R}$, consider the subgroup generated by a nonelementary discrete isometry group of $\mathbb{H}^2_\mathbb{R}$ and the hyperbolic reflexion of $\mathbb{H}^3_\mathbb{R}$ with fixed point set $\mathbb{H}^2_\mathbb{R}$). But the pointwise stabiliser $\operatorname{Fix}_\Gamma(\Lambda\Gamma)$ of $\Lambda\Gamma$ in Γ is a finite normal subgroup of Γ.

Lemma 2.1 *The set of fixed points in $\Lambda\Gamma$ of any element γ of $\Gamma - \mathrm{Fix}_\Gamma(\Lambda\Gamma)$ is a closed subset with empty interior in $\Lambda\Gamma$.*

Proof. This is well known, but difficult to locate in a reference. We may assume γ to be elliptic. If γ fixes a fixed point of a hyperbolic element α, then γ pointwise fixes its translation axis Axe_α: otherwise, $(\alpha^{-n}\gamma\alpha^n)_{n\in\mathbb{N}}$ is a sequence of pairwise distinct elements of Γ mapping any given point in Axe_α at bounded distance from itself, which contradicts the discreteness of Γ. Since the set of pairs of endpoints of the translation axes of hyperbolic elements of Γ is dense in $\Lambda\Gamma \times \Lambda\Gamma$, if γ pointwise fixes an open subset of $\Lambda\Gamma$, then γ is the identity on $\Lambda\Gamma$. □

In particular, by Baire's theorem, the union of the sets of fixed points in $\Lambda\Gamma$ of the elements of $\Gamma - \mathrm{Fix}_\Gamma(\Lambda\Gamma)$ is a Γ-invariant Borel subset, whose complement is a dense G_δ-set.

The *Busemann cocycle* is the map $\beta : \partial_\infty\widetilde{M} \times \widetilde{M} \times \widetilde{M} \to \mathbb{R}$, defined by

$$(\xi, x, y) \mapsto \beta_\xi(x, y) = \lim_{t\to+\infty} d(x, \xi_t) - d(y, \xi_t) \;,$$

where $t \mapsto \xi_t$ is any geodesic ray ending at ξ. For every $\xi \in \partial_\infty\widetilde{M}$, the *horospheres centered at* ξ are the level sets of the map $y \mapsto \beta_\xi(y, x_0)$ from $\widetilde{M}$ to $\mathbb{R}$, and the (closed) *horoballs centered at* ξ are its sublevel sets, for any $x_0 \in \widetilde{M}$.

2.3 The geometry of the unit tangent bundle

We denote by $\pi : T^1\widetilde{M} \to \widetilde{M}$ the unit tangent bundle of $\widetilde{M}$, and we endow $T^1\widetilde{M}$ with Sasaki's Riemannian metric (see below). We denote by $\iota : T^1\widetilde{M} \to T^1\widetilde{M}$ the flip map $v \mapsto -v$.

Consider the Riemannian orbifold $M = \Gamma\backslash\widetilde{M}$ and let us denote by T^1M and TT^1M the quotient Riemannian orbifolds $\Gamma\backslash T^1\widetilde{M}$ and $\Gamma\backslash TT^1\widetilde{M}$. We denote by $F : T^1M \to \mathbb{R}$ and again by $\iota : T^1M \to T^1M$ and $\pi : T^1M \to M$ the quotient maps of $\widetilde{F}$, ι and π.

For every $v \in T^1\widetilde{M}$, we denote by v_- and v_+ the points at $-\infty$ and $+\infty$ of the geodesic line $\ell_v : \mathbb{R} \to \widetilde{M}$ with $\dot\ell_v(0) = v$; we have $(\iota v)_\pm = v_\mp$.

We denote by $\widetilde{\Omega}\Gamma$ (respectively $\widetilde{\Omega}_c\Gamma$) the closed (respectively Borel) Γ-invariant set of elements $v \in T^1\widetilde{M}$ such that v_- and v_+ both belong to $\Lambda\Gamma$ (respectively $\Lambda_c\Gamma$), and by $\Omega\Gamma$ and $\Omega_c\Gamma$ their images by the canonical projection $T^1\widetilde{M} \to T^1M$. The union of the sets of fixed points in $\widetilde{\Omega}\Gamma$ of the elements of $\Gamma - \mathrm{Fix}_\Gamma(\Lambda\Gamma)$ is properly contained in $\widetilde{\Omega}\Gamma$.

For every $x \in \widetilde{M}$, let $\iota_x : \partial_\infty\widetilde{M} \to \partial_\infty\widetilde{M}$ be the *antipodal map with respect to* x, that is the involutive map $\xi \mapsto v_-$ where v is the unique element of $T^1\widetilde{M}$ such that $\pi(v) = x$ and $v_+ = \xi$. Note that for every isometry γ of $\widetilde{M}$ and for every x in $\widetilde{M}$, we have

$$\iota_{\gamma x} = \gamma \circ \iota_x \circ \gamma^{-1} \;.$$

Let us now recall a parametrisation of $T^1\widetilde{M}$ in terms of the boundary at infinity of $\widetilde{M}$. Let $\partial^2_\infty\widetilde{M} = (\partial_\infty\widetilde{M} \times \partial_\infty\widetilde{M}) - \Delta$, where Δ is the diagonal in $\partial_\infty\widetilde{M} \times \partial_\infty\widetilde{M}$. For every base point x_0 in $\widetilde{M}$, the space $T^1\widetilde{M}$ may be identified with $\partial^2_\infty\widetilde{M} \times \mathbb{R}$, by the map which maps a unit tangent vector v to the triple (v_-, v_+, t) where t is the algebraic distance on $\ell_v(\mathbb{R})$ (oriented from v_- to v_+) between $\ell_v(0)$ and the closest point of $\ell_v(\mathbb{R})$ to x_0.

This parametrisation (called the *Hopf parametrisation* defined by x_0) differs from the one defined by another base point x_0' only by an additive term on the third factor (independent of the time t).

We end this subsection by giving some information on the Sasaki metric on $T\widetilde{M}$ (see for instance [Bal, Chap. IV]). Recall that it has the following defining properties: The vector bundle $TT\widetilde{M} \to T\widetilde{M}$ has a Whitney sum decomposition $TT\widetilde{M} = V \oplus H$ such that, for every $v \in T\widetilde{M}$, the direct sum decomposition of the fibers

$$T_vT\widetilde{M} = V_v \oplus H_v$$

is orthogonal for Sasaki's scalar product, and if $\pi : T\widetilde{M} \to \widetilde{M}$ also denotes the full tangent bundle oof $\widetilde{M}$,

- $T\pi_{|H_v} : H_v \to T_{\pi(v)}\widetilde{M}$ is an isometric linear isomorphism,
- $V_v = \operatorname{Ker} T_v\pi = T_v(T_{\pi(v)}\widetilde{M}) = T_{\pi(v)}\widetilde{M}$ (equality as Euclidean spaces),
- if ∇ is the Levi-Civita connection of $\widetilde{M}$, if $p_V : TT\widetilde{M} \to V$ is the bundle map which is the linear projection from $T_vT\widetilde{M}$ on V_v parallel to H_v for every $v \in T\widetilde{M}$, then for every vector field $X : \widetilde{M} \to T\widetilde{M}$ on $\widetilde{M}$, with $TX : T\widetilde{M} \to TT\widetilde{M}$ its tangent map, we have

$$\nabla_v X = p_V \circ TX(v)\ .$$

The map $\pi : T^1\widetilde{M} \to \widetilde{M}$ is a Riemannian submersion with totally geodesic fibers, and if γ is an isometry of $\widetilde{M}$, then its tangent map, still denoted by γ, is an isometry of $T^1\widetilde{M}$. The flip map ι is also an isometry of the Sasaki metric. The Riemannian distance d_S induced by the Sasaki metric satisfies (see [Sol] for generalisations)

$$d_S(v, w) = \inf_\alpha \sqrt{\ell(\alpha)^2 + \|\ (v - \|_\alpha w)\ \|}\ , \tag{3}$$

where the lower bound is taken on the smooth paths $\alpha : [0, 1] \to \widetilde{M}$ of length $\ell(\alpha)$ from $\pi(w)$ to $\pi(v)$ and $\|_\alpha$ is the parallel transport along α. Note that for all $v, w \in T^1\widetilde{M}$

$$d_S(v, w) \geq d(\pi(v), \pi(w))\ . \tag{4}$$

We will endow $T^1\widetilde{M}$ with its quite usual distance $d = d_{T^1\widetilde{M}}$, defined as follows: for all $v, v' \in T^1\widetilde{M}$,

$$d(v, v') = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} d(\pi(\phi_t v), \pi(\phi_t v'))\, e^{-t^2}\ dt\ . \tag{5}$$

Note that for all $v \in T^1\widetilde{M}$ and $t \in \mathbb{R}$, we have

$$d(v, \phi_t v) = |t|\ ,$$

and for all $v, v' \in T^1\widetilde{M}$ and $\gamma \in \operatorname{Isom}(\widetilde{M})$, we have

$$d(\gamma\, v, \gamma\, v') = d(v, v') \quad \text{and} \quad d(\iota\, v, \iota\, v') = d(v, v')\ . \tag{6}$$

We will sometimes consider another distance $d' = d'_{T^1\widetilde{M}}$ on $T^1\widetilde{M}$, defined by (using the convexity of the distance in $\widetilde{M}$ to obtain the second equality)

$$\begin{aligned}\forall\, v, w \in T^1\widetilde{M},\quad d'_{T^1\widetilde{M}}(v, w) &= \max_{s\in[-1,0]} d_{\widetilde{M}}\big(\pi(\phi_s v), \pi(\phi_s w)\big) \\ &= \max\big\{d_{\widetilde{M}}\big(\pi(v), \pi(w)\big), d_{\widetilde{M}}\big(\pi(\phi_{-1} v), \pi(\phi_{-1} w)\big)\big\}\ .\end{aligned} \tag{7}$$

The distance d' is (Lipschitz-)equivalent to the Riemannian distance d_S induced by the Sasaki metric on $T^1\widetilde{M}$, since $\widetilde{M}$ has pinched negative sectional curvature (see for instance [Bal, page 70]). Note that the flip map ι, which is an isometry of the Sasaki metric, is hence Lipschitz for this metric, with Lipschitz constant depending only on the bounds on the sectional curvature.

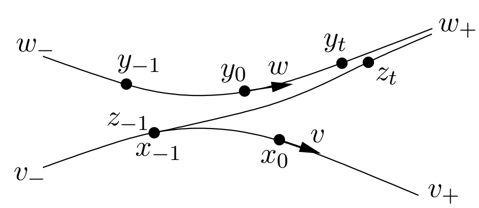
The following result should be well known, but we provide a proof in the absence of a precise reference.

Lemma 2.2 *The distance d on $T^1\widetilde{M}$ is Hölder-equivalent to the Riemannian distance d_S induced by Sasaki's Riemannian metric.*

In particular, when we will consider Hölder-continuous functions $\widetilde{F} : T^1\widetilde{M} \to \mathbb{R}$, we may use any one of the three distances d_S, $d = d_{T^1\widetilde{M}}$ and $d' = d'_{T^1\widetilde{M}}$, remembering that the Hölder constants depend on this distance.

Proof. It is sufficient to prove that the distances d and d' on $T^1\widetilde{M}$ are Hölder-equivalent.

Let $v, w \in T^1\widetilde{M}$. Define $x_t = \pi(\phi_t v)$ and $y_t = \pi(\phi_t w)$ for every $t \in \mathbb{R}$, as well as $\epsilon_0 = d(x_0, y_0)$ and $\epsilon_{-1} = d(x_{-1}, y_{-1})$. Let $t \mapsto z_t$ be the geodesic ray parametrised by $[-1, +\infty[$ from x_{-1} to the point at infinity w_+.



By the definition of d', for every $t \in [-1, 0]$, we have

$$d(x_t, y_t) \leq d'(v, w) = \max\{\epsilon_0, \epsilon_{-1}\} .$$

By convexity, for every $t \geq -1$, we have $d(z_t, y_t) \leq \epsilon_{-1}$. Since $\widetilde{M}$ has a finite lower bound on its curvature, there exists a constant $c_1 > 0$ such that for every $t \geq 0$, we have $d(x_t, z_t) \leq d(x_0, z_0)\ e^{c_1 t}$. Hence, by the triangle inequality, for every $t \geq 0$, we have

$$d(x_t, y_t) \leq \epsilon_{-1} + \epsilon_0\ e^{c_1 t} .$$

Similarly, considering the geodesic ray from x_0 to w_-, for every $t \leq -1$, we have $d(x_t, y_t) \leq \epsilon_0 + \epsilon_{-1}\ e^{c_1 (1-t)}$. Therefore $\sqrt{\pi}\ d(v, w)$ is bounded from above by

$$\int_{-\infty}^{-1} (\epsilon_0 + \epsilon_{-1}\ e^{c_1 (1-t)})\ e^{-t^2}\ dt + \int_{-1}^{0} \max\{\epsilon_0, \epsilon_{-1}\}\ e^{-t^2}\ dt + \int_{0}^{+\infty} (\epsilon_{-1} + \epsilon_0\ e^{c_1 t})\ e^{-t^2}\ dt ,$$

and there exists a constant $c_2 > 0$ such that $d(v, w) \leq c_2\ (\epsilon_{-1} + \epsilon_0) \leq 2\, c_2\ d'(v, w)$.

Conversely, assume that $d'(v, w) \leq 1$. Then $\epsilon_0, \epsilon_1 \leq 1$ and

$$\begin{aligned}
\sqrt{\pi}\ d(v, w) &\geq \int_0^{\epsilon_0/4} d(x_t, y_t)\ e^{-t^2}\ dt + \int_{-1}^{-1+\epsilon_{-1}/4} d(x_t, y_t)\ e^{-t^2}\ dt \\
&\geq \int_0^{\epsilon_0/4} (\epsilon_0 - 2t)\ e^{-t^2}\ dt + \int_{-1}^{-1+\epsilon_{-1}/4} (\epsilon_{-1} - 2(t+1))\ e^{-t^2}\ dt \\
&\geq \frac{\epsilon_0}{4} \times \frac{\epsilon_0}{2} \times \frac{1}{e}\ +\ \frac{\epsilon_{-1}}{4} \times \frac{\epsilon_{-1}}{2} \times \frac{1}{e} \geq \frac{1}{16\, e}(\epsilon_0 + \epsilon_{-1})^2 \geq \frac{1}{16\, e}(d'(v, w))^2 .
\end{aligned}$$

This proves the result. $\square$

2.4 Geodesic flow, (un)stable foliations and the Hamenstädt distances

We denote by $(\phi_t : T^1\widetilde{M} \to T^1\widetilde{M})_{t\in\mathbb{R}}$ the geodesic flow of $\widetilde{M}$, which satisfies $\iota\circ\phi_t = \phi_{-t}\circ\iota$. In the Hopf parametrisation, the geodesic flow is the action of $\mathbb{R}$ by translation on the third factor. We denote again by $\phi_t : T^1M \to T^1M$ the quotient map of ϕ_t.

Given $x \neq y$ in $\widetilde{M}$, the unit tangent vector at x *pointing towards* y is the unique element $v \in T^1_x\widetilde{M}$ such that there exists $t > 0$ with $\pi(\phi_t v) = y$.

By [Ebe, Coro. 3.8], the closed subset $\Omega\Gamma$ defined in the previous subsection is the (topological) *nonwandering set* of the geodesic flow in T^1M, that is, the set of points $v \in T^1M$ such that, for every neighbourhood U of v, there exists a sequence $(t_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ converging to $+\infty$ such that, for every $n \in \mathbb{N}$, the intersection $U \cap \phi_{t_n}U$ is nonempty. The support of any Borel probability measure on T^1M which is invariant under the geodesic flow is contained in $\Omega\Gamma$. Note that $\Omega_c\Gamma$ is dense in $\Omega\Gamma$, and is the *two-sided recurrent set* of the geodesic flow in T^1M, that is, the set of unit tangent vectors whose geodesic line comes back in some compact subset of M at times tending to $+\infty$ and to $-\infty$.

We now recall the definitions and properties of the dynamical foliations associated to the geodesic flow on $T^1\widetilde{M}$. For every $v \in T^1\widetilde{M}$, we define the *strong stable leaf* of v, *strong unstable leaf* of v, *(weak or central) stable leaf* of v and *(weak or central) unstable leaf* of v respectively as

$$W^{ss}(v) = \{w \in T^1\widetilde{M} \;:\; \lim_{t\to+\infty} d(\phi_t v, \phi_t w) = 0\}\;,$$

$$W^{su}(v) = \{w \in T^1\widetilde{M} \;:\; \lim_{t\to-\infty} d(\phi_t v, \phi_t w) = 0\}\;,$$

$$W^{s}(v) = \{w \in T^1\widetilde{M} \;:\; \exists s \in \mathbb{R},\; \lim_{t\to+\infty} d(\phi_{t+s} v, \phi_t w) = 0\}\;,$$

$$W^{u}(v) = \{w \in T^1\widetilde{M} \;:\; \exists s \in \mathbb{R},\; \lim_{t\to-\infty} d(\phi_{t+s} v, \phi_t w) = 0\}\;.$$

These four families define continuous foliations of $T^1\widetilde{M}$ with smooth leaves, denoted by $\widetilde{\mathscr{W}}^{ss}$, $\widetilde{\mathscr{W}}^{su}$, $\widetilde{\mathscr{W}}^{s}$, $\widetilde{\mathscr{W}}^{u}$ and called the *strong stable, strong unstable, stable, unstable foliations* of the geodesic flow $(\phi_t)_{t\in\mathbb{R}}$ on $T^1\widetilde{M}$.

The smooth submanifolds $\pi(W^{ss}(v))$ and $\pi(W^{su}(v))$ of $\widetilde{M}$ are the horospheres passing through $\pi(v)$ with centers respectively v_+ and v_-, and conversely $W^{ss}(v)$ and $W^{su}(v)$ are the subsets of $T^1\widetilde{M}$ consisting respectively of the inner and outer unit normal vectors to these two horospheres. Note that $W^s(v)$ and $W^u(v)$ are invariant under the geodesic flow $(\phi_t)_{t\in\mathbb{R}}$. Furthermore, for every $v \in T^1\widetilde{M}$, the map $w \mapsto w_+$ from $W^{su}(v)$ to $\partial_\infty\widetilde{M} - \{v_-\}$ is a homeomorphism, and the map $w \mapsto w_-$ from $W^{ss}(v)$ to $\partial_\infty\widetilde{M} - \{v_+\}$ is also a homeomorphism.

For every $v \in T^1M$, denote again by $W^{ss}(v)$ (respectively $W^{su}(v)$, $W^s(v)$, $W^u(v)$) the image by the canonical projection $T^1\widetilde{M} \to T^1M$ of $W^{ss}(\widetilde{v})$ (respectively $W^{su}(\widetilde{v})$, $W^s(\widetilde{v})$, $W^u(\widetilde{v})$), where $\widetilde{v}$ is any lift of v to $T^1\widetilde{M}$. Note that $W^{ss}(\iota v) = \iota(W^{su}(v))$, $W^{su}(\iota v) = \iota(W^{ss}(v))$, $W^s(\iota v) = \iota(W^u(v))$, $W^u(\iota v) = \iota(W^s(v))$ for every $v \in T^1\widetilde{M}$ or $v \in T^1M$. The sets $W^{ss}(v)$ (respectively $W^{su}(v)$, $W^s(v)$, $W^u(v)$) define a partition of T^1M, denoted by $\mathscr{W}^{ss}$ (respectively $\mathscr{W}^{su}$, $\mathscr{W}^{s}$, $\mathscr{W}^{u}$), which is a foliation outside the image in T^1M of the set of fixed points of elements of $\Gamma - \{\mathrm{id}\}$ in $T^1\widetilde{M}$.

Let us also recall a natural family of distances on the strong stable leaves of $T^1\widetilde{M}$ (see for instance [HeP1, Appendix], slightly modifying the definition of [Ham1], as well as

[HeP2, §2.2] for a generalisation when a horosphere is replaced by the boundary of any nonempty closed convex subset). For every $w \in T^1\widetilde{M}$, let $d_{W^{\rm su}(w)}$ be the *Hamenstädt distance* on the strong unstable leaf of w, defined as follows: for all $v, v' \in W^{\rm su}(w)$,

$$d_{W^{\rm su}(w)}(v, v') = \lim_{t\to+\infty} e^{\frac{1}{2}d(\pi(\phi_t v),\, \pi(\phi_t v'))-t} \;.$$

This limit exists, and the Hamenstädt distance is a distance inducing the original topology on $W^{\rm su}(w)$. For all $v, v' \in W^{\rm su}(w)$ and for every isometry γ of $\widetilde{M}$, we have

$$d_{W^{\rm su}(\gamma w)}(\gamma v, \gamma v') = d_{W^{\rm su}(w)}(v, v') \;.$$

For all $w \in T^1\widetilde{M}$, $s \in \mathbb{R}$ and $v, v' \in W^{\rm su}(w)$, we have

$$d_{W^{\rm su}(\phi_s w)}(\phi_s v, \phi_s v') = e^s \; d_{W^{\rm su}(w)}(v, v') \;. \tag{8}$$

The following lemma compares the Hamenstädt distance $d_{W^{\rm su}(w)}$ with the distance d on $\widetilde{M}$ and the distance d on $T^1\widetilde{M}$.

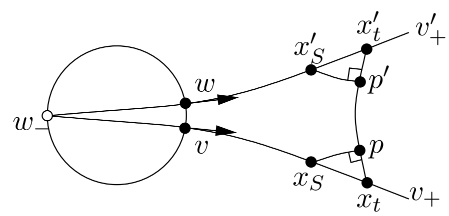
Lemma 2.3 *There exists $c > 0$ such that, for all $w \in T^1\widetilde{M}$ and $v, v' \in W^{\rm su}(w)$, we have*

$$\max\{\; \frac{1}{c}\; d(v, v'),\; d(\pi(v), \pi(v'))\;\} \le d_{W^{\rm su}(w)}(v, v') \le e^{\frac{1}{2}d(\pi(v),\, \pi(v'))} \;. \tag{9}$$

Proof. We give proofs only for the sake of completeness. The inequality on the right hand side follows from the triangle inequality.

The inequality $d(v, v') \le c\; d_{W^{\rm su}(w)}(v, v')$ for some universal constant $c > 0$ is due to [PaP2, Lem. 3], as follows.

We may assume that $v \neq v'$. Let $x_t = \pi(\phi_t v)$ and $x'_t = \pi(\phi_t v')$. By the convexity properties of the distance in $\widetilde{M}$, the map from $\mathbb{R}$ to $\mathbb{R}$ defined by $t \mapsto d(x_t, x'_t)$ is increasing, with image $]0, +\infty[$. Let $S \in \mathbb{R}$ be such that $d(x_S, x'_S) = 1$. For every $t \ge S$, let p and p' be the closest point projections of x_S and x'_S respectively on the geodesic segment $[x_t, x'_t]$.



In the hyperbolic upper halfplane $\mathbb{H}^2_\mathbb{R}$, the points i, $1-2i$ and $1+2i$ are pairwise at distance $2\log\frac{1+\sqrt{5}}{2} < 1$. Hence by convexity, if x, y are the points of $\mathbb{H}^2_\mathbb{R}$ with horizontal coordinates $1, -1$ respectively, with same vertical coordinate and at hyperbolic distance 1, then the distance from x to the geodesic line with endpoints $1, -1$ is at most 1. By comparison, we therefore have $d(p, x_S),\; d(p', x'_S) \le 1$. Hence, by convexity and the triangle inequality,

$$\begin{aligned} d(x_t, x'_t) &\ge d(x_t, p) + d(p', x'_t) \\ &\ge d(x_t, x_S) - 1 + d(x'_t, x'_S) - 1 = 2(t - S - 1) \;. \end{aligned}$$

Thus by the definition of the Hamenstädt distance $d_{W^{\rm su}(w)}$, we have

$$d_{W^{\rm su}(w)}(v, v') \ge e^{-S-1} \;. \tag{10}$$

By the triangle inequality, if $t \geq S$, then

$$d(x_t, x'_t) \leq d(x_t, x_S) + d(x_S, x'_S) + d(x'_S, x'_t) = 2(t-S) + 1 \ .$$

Since $\widetilde{M}$ is CAT(-1), if $t \leq S$, we have by comparison

$$d(x_t, x'_t) \leq e^{t-S}\, d(x_S, x'_S) = e^{t-S} \ .$$

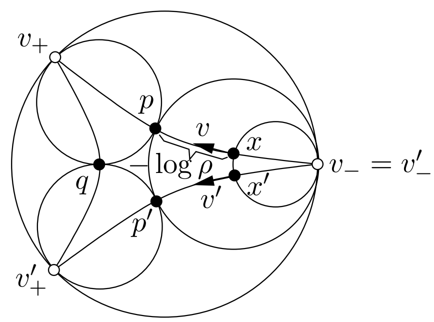
Therefore, by the definition of the distance d on $T^1\widetilde{M}$ (see Equation (5)), we have

$$d(v, v') \leq \int_{-\infty}^{S} e^{t-S}\, e^{-t^2}\, dt + \int_{S}^{+\infty} (2(t-S)+1)\, e^{-t^2}\, dt = \mathrm{O}(e^{-S}) \ .$$

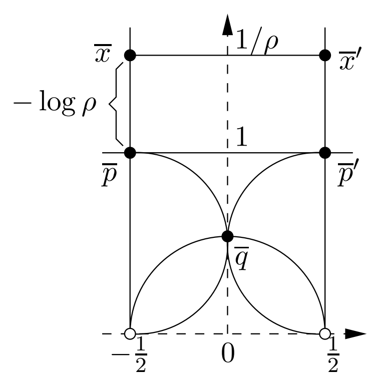
The result hence follows from Equation (10).

The last inequality $d(\pi(v), \pi(v')) \leq d_{W^{\rm su}(w)}(v, v')$ is due to [PaP4, §2], as follows.

Let $x = \pi(v)$, $x' = \pi(v')$ and $\rho = d_{W^{\rm su}(w)}(v, v')$. Consider the ideal triangle Δ with vertices v_+, v'_+ and $v_- = v'_-$. Let $p \in\,]v_-, v_+[\,$, $p' \in\,]v'_-, v'_+[\,$ and $q \in\,]v_+, v'_+[\,$ be the pairwise tangency points of horospheres centered at the vertices of Δ: $\beta_{v_-}(p, p') = 0$, $\beta_{v_+}(p, q) = 0$ and $\beta_{v'_+}(p', q) = 0$. By definition of the Hamenstädt distance, the algebraic distance from x to p on the geodesic line $]v_-, v_+[$ (oriented from v_- to v_+) is $-\log \rho$.



Consider the ideal triangle $\overline{\Delta}$ in the hyperbolic upper halfplane $\mathbb{H}^2_{\mathbb{R}}$, with vertices $-\frac{1}{2}$, $\frac{1}{2}$ and ∞. Let $\overline{p} = (-\frac{1}{2}, 1)$, $\overline{p}' = (\frac{1}{2}, 1)$ and $\overline{q} = (0, \frac{1}{2})$ be the pairwise tangency points of horospheres centered at the vertices of $\overline{\Delta}$. Let $\overline{x}$ and $\overline{x}'$ be the point at algebraic (hyperbolic) distance $-\log \rho$ from $\overline{p}$ and $\overline{p}'$, respectively, on the upwards oriented vertical line through them. By comparison, we have $d(x, x') \leq d(\overline{x}, \overline{x}') \leq 1/e^{-\log \rho} = \rho$. $\square$



We define analogously the *Hamenstädt distance* $d_{W^{\rm ss}(w)}$ on the strong stable leaf of $w \in T^1\widetilde{M}$ by

$$d_{W^{\rm ss}(w)}(v, v') = \lim_{t \to +\infty} e^{\frac{1}{2} d(\pi(\phi_{-t} v),\, \pi(\phi_{-t} v')) - t} \ .$$

It satisfies analogous properties, in particular that for all $w \in T^1\widetilde{M}$, $s \in \mathbb{R}$ and $v, v' \in W^{\rm ss}(w)$, we have

$$d_{W^{\rm ss}(\phi_s w)}(\phi_s v, \phi_s v') = e^{-s}\, d_{W^{\rm ss}(w)}(v, v') \ . \tag{11}$$

The Hamenstädt distances on the strong stable and strong unstable leaves are related as follows: for all $w \in T^1\widetilde{M}$ and $v, v' \in W^{su}(w)$, we have

$$d_{W^{\rm ss}(\iota\, w)}(\iota\, v, \iota\, v') = d_{W^{\rm su}(w)}(v, v') \ . \tag{12}$$

2.5 Some exercises in hyperbolic geometry

We end this chapter by the following (well known, see for instance [PaP1]) exercises in hyperbolic geometry.

Lemma 2.4 *(i) For all x, y, z in $\widetilde{M}$ such that $d(x, z) = d(y, z)$, for every $t \in [0, d(x, z)]$, if x_t (respectively y_t) is the point on $[x, z]$ (respectively $[y, z]$) at distance t from x (respectively y), then*

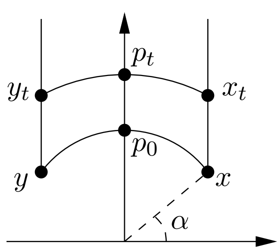
$$d(x_t, y_t) \leq e^{-t} \sinh d(x, y) \;.$$

(ii) For every $\alpha > 0$, there exists $\beta > 0$ (depending only on α and the bounds on the sectional curvature of $\widetilde{M}$) such that for all x, y, z in $\widetilde{M}$ such that $d(x, z) = d(y, z) \geq 1$ and $d(x, y) \leq \alpha$, for every $t \in [0, d(x, z)]$, with x_t and y_t defined as above,

$$d(x_t, y_t) \leq \beta\, d(x_1, y_1)\, e^{-t} \;.$$

Proof. (i) We may assume that $x \neq y$, otherwise $x_t = y_t$ and the result is trivially true. Since the sectional curvature of $\widetilde{M}$ is at most -1, and by comparison, we may assume that $\widetilde{M}$ is the hyperbolic upper halfspace $\mathbb{H}^2_{\mathbb{R}}$. Let m be the orthogonal projection of z on the geodesic line through x and y, which is the midpoint of $[x, y]$ by symmetry.

Replacing z by the point at infinity of the geodesic ray starting from m and passing through z (any one perpendicular to $[x, y]$ if $m = z$) increases $d(x_t, y_t)$ and does not change $d(x, y)$. Hence we may assume that z is the point at infinity in $\mathbb{H}^2_{\mathbb{R}}$, and that x and y lie on the (Euclidean) circle of center 0 and radius 1, have same (Euclidean) height, with the real part of x positive, the real part of y negative.



Let p_t be the intersection point of $[x_t, y_t]$ with the vertical axis, which is by symmetry the midpoint of $[x_t, y_t]$. If α is the (Euclidean) angle at 0 between the horizontal axis and the (Euclidean) line from 0 passing through x, then an easy computation in hyperbolic geometry (see also [Bea], page 145) gives $\sinh d(x, p_0) = \cos\alpha / \sin\alpha$.

Similarly, $\sinh d(x_t, p_t) = \cos\alpha/(e^t \sin\alpha)$. So that

$$d(x_t, y_t) = 2\, d(x_t, p_t) \leq 2 \sinh d(x_t, p_t) = 2\, e^{-t} \sinh d(x, p_0) = 2\, e^{-t} \sinh \frac{d(x, y)}{2} \;,$$

which proves the first result, as $2 \sinh \frac{t}{2} \leq \sinh t$ if $t \geq 0$.

(ii) Observe first that for all $a > 0$ and $s \in [0, a]$, we have $s \leq \sinh s \leq s\, \frac{\sinh a}{a}$. By (i), we only have to prove that there exists a constant $c > 0$ such that $d(x_1, y_1) \geq c\, d(x, y)$. By comparison, we may assume that $\widetilde{M}$ is the real hyperbolic plane with constant curvature $-b^2$. The above computations give $\sinh \frac{d(x_1, y_1)}{2} = e^{-1} \sinh \frac{d(x, y)}{2}$, hence the result follows from the preliminary remark. $\Box$

Corollary 2.5 *(i) For every $T \in [0, +\infty[$, for all v and w in $T^1\widetilde{M}$ such that $\pi(\phi_T v) = \pi(\phi_T w)$, for every $t \in [0, T]$,*

$$d'_{T^1\widetilde{M}}(\phi_t v, \phi_t w) \leq e^{-t} \sinh\big(2 + d(\pi(v), \pi(w))\big) \;.$$

(ii) For every $\alpha' > 0$, there exists $\beta' > 0$ (depending only on α' and the bounds on the sectional curvature of $\widetilde{M}$) such that for every $T \in [1, +\infty[$, for all v and w in $T^1\widetilde{M}$ such that $\pi(\phi_T v) = \pi(\phi_T w)$ and $d(\pi(v), \pi(w)) \leq \alpha'$, for every $t \in [0, T]$,

$$d'_{T^1\widetilde{M}}(\phi_t v, \phi_t w) \leq \beta' \, d(\pi(v), \pi(w))\, e^{-t} \, .$$

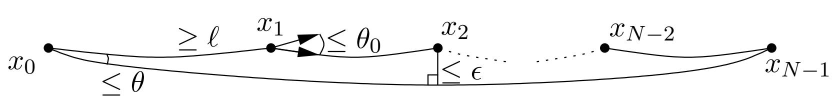
Proof. (i) By the definition of the distance $d'_{T^1\widetilde{M}}$ (see above Lemma 2.2), this follows from Lemma 2.4 (i), applied to $x = \pi(\phi_{-1}v)$, $y = \pi(\phi_{-1}w)$ and $z = \pi(\phi_T v)$, since $d(x, y) \leq d(\pi(v), \pi(w)) + 2$.

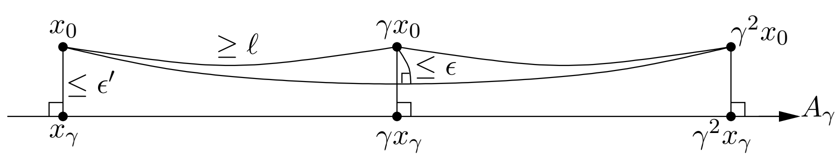
(ii) This follows similarly from Lemma 2.4 (ii), with $\alpha = \alpha' + 2$. $\square$

Also recall without proof the following well known results in complete simply connected Riemannian manifolds with sectional curvature at most -1 (variants of Anosov's closing lemma).

Lemma 2.6 *For all $\ell, \epsilon, \theta > 0$, there exists $\theta_0 = \theta_0(\ell, \epsilon, \theta) > 0$ such that for every piecewise geodesic path $\omega = \bigcup_{0 \leq i < N} [x_i, x_{i+1}]$ in $\widetilde{M}$ with $N \in \mathbb{N} \cup \{+\infty\}$, if its exterior angle at x_i is at most θ_0 for $0 < i < N - 1$ and if the length of each segment $[x_i, x_{i+1}]$ is at least ℓ, then*

- *ω is contained in the ε-neighbourhood of the geodesic segment $\omega' = [x_0, x_N]$, if N is finite, or of the geodesic ray $\omega' = [x_0, \xi]$ where $\xi \in \partial_\infty \widetilde{M}$ is the point at infinity to which the sequence $(x_i)_{i \in \mathbb{N}}$ converges, if N is infinite;*
- *the angles between ω and ω' at the finite endpoints of ω' are at most θ.*





Lemma 2.7 *For all $\ell, \epsilon' > 0$, there exists $\epsilon = \epsilon(\ell, \epsilon') \in \,]0, 1]$ with $\lim_{\epsilon' \to 0} \epsilon = 0$ such that for every isometry γ of any proper geodesic CAT(-1)-space X, for every x_0 in X, if $d(x_0, \gamma x_0) \geq \ell$ and $d(\gamma x_0, [x_0, \gamma^2 x_0]) \leq \epsilon$, then γ is hyperbolic and $d(x_0, Axe_\gamma) \leq \epsilon'$ where Axe_γ is the translation axis of γ in X.*

2.6 Pushing measures by branched covers

In this text, we will often construct measures on T^1M starting from Γ-invariant measures on $T^1\widetilde{M}$. Given a Galois covering $f : X \to Y$ of topological spaces, and a (positive Borel) measure m on X invariant under the covering group, it is well known that there exists a unique measure on Y (which is not the pushforward measure $f_* m$) such that the map f locally preserves the measure. In the case of ramified covers, the analogous construction is not so well known (see for instance [PaP4, §2.4]). The fact that we are not assuming Γ to be torsion free requires it.

Let $\widetilde{X}$ be a locally compact metrisable space, endowed with a proper (but not necessarily free) action of a discrete group G. Let $p : \widetilde{X} \to X = G\backslash\widetilde{X}$ be the canonical projection. Let $\widetilde{\mu}$ be a locally finite G-invariant measure.

Note that the map N from $\widetilde{X}$ to $\mathbb{N} - \{0\}$ sending a point $x \in X$ to the order of its stabiliser in G is upper semicontinuous. In particular, for every $n \geq 1$, the G-invariant subset $\widetilde{X}_n = N^{-1}(\{n\})$ is locally closed, hence locally compact metrisable. With $X_n = p(\widetilde{X}_n)$, the restriction $p_{|\widetilde{X}_n} : \widetilde{X}_n \to X_n$ is a local homeomorphism. Since $\widetilde{\mu}$ is G-invariant, there exists a unique measure μ_n on X_n such that the map $p_{|\widetilde{X}_n}$ locally preserves the measure. Now, considering a measure on X_n as a measure on X with support in X_n, define

$$\mu = \sum_{n \geq 1} \frac{1}{n}\, \mu_n\ ,$$

which is a locally finite measure on X, called the measure *induced by* $\widetilde{\mu}$ on X.

Note that if $\widetilde{\mu}$ gives measure 0 to the set $N^{-1}([2, +\infty[)$ of fixed points of non-trivial elements of G, then $\mu = \mu_1$, and the above construction is not really needed.

If $\widetilde{X}'$ is another locally compact metrisable space, endowed with a proper action of a discrete group G', if $\widetilde{\mu}'$ is a locally finite G'-invariant measure on $\widetilde{X}'$, with induced measure μ' on $G'\backslash X'$, then the measure on the product of the quotient spaces $G\backslash X \ \times\ G'\backslash X'$, induced by the product measure $\widetilde{\mu} \otimes \widetilde{\mu}'$, is the product $\mu \otimes \mu'$ of the induced measures.

It is easy to check that the map $\widetilde{\mu} \mapsto \mu$ from the space of locally finite measures on $\widetilde{X}$ to the space of locally finite measures on X, both endowed with their weak star topologies, is continuous.

3 A Patterson-Sullivan theory for Gibbs states

Let $(\widetilde{M}, \Gamma, \widetilde{F})$ be as in the beginning of Chapter 2: $\widetilde{M}$ is a complete simply connected Riemannian manifold, with dimension at least 2 and pinched sectional curvature at most -1; Γ is a nonelementary discrete group of isometries of $\widetilde{M}$; and $\widetilde{F} : T^1\widetilde{M} \to \mathbb{R}$ is a Hölder-continuous Γ-invariant map.

We recall in this chapter the construction of the Gibbs measure on the unit tangent bundle of a negatively curved manifold associated with a given potential, due to O. Mohsen [Moh]. More precisely, the subsections 3.4, 3.5, 3.7, 3.8, 3.9 are new (except from the strong influence of [Ham2]), the others are extracted (up to an adaptation to the noncocompact case) from [Moh]. We refer to [Ham2, Led2, Cou2, Sch3] for different approaches. The Gibbs cocycles used in these four references are the same up to signs as the one we define in Subsection 3.3. Note that the last three references use $\sum_{\gamma\in\Gamma} e^{s\int_x^{\gamma y} \widetilde{F}}$ for their Poincaré series, whereas we use $\sum_{\gamma\in\Gamma} e^{\int_x^{\gamma y}(\widetilde{F}-s)}$. This additive instead of multiplicative approach greatly simplifies the techniques.

3.1 Potential functions and their periods

For all x, y in $\widetilde{M}$, define

$$\int_x^y \widetilde{F} = \int_0^{d(x,\,y)} \widetilde{F}(\phi_t(v))\ dt\ ,$$

where v is a unit tangent vector such that $\pi(v) = x$ and $\pi(\phi_{d(x,y)}v) = y$ (unique if $y \neq x$).

Remark. A general extension of the techniques of this work when $\widetilde{M}$ is any complete locally compact CAT(-1) metric space as in [Rob1] (even assuming that its geodesic segments are extendible to geodesic lines) seems difficult at this point. The correct analog of the unit tangent bundle for such an extension is probably not the usual one of the space of all geodesic lines. Indeed, there could be many geodesic lines containing a given segment $[x,y]$. Hence defining $\int_x^y \widetilde{F}$ would require a choice of one of these geodesic lines, or restrictions on $\widetilde{F}$ or some averaging process, hence appropriate additional information in order not to lose the invariance under the group Γ. A better analog would be the space of germs of geodesic segments, but the geodesic flow then does not extend. See the case of trees in [BrPP].

Note that

$$\forall\, \gamma\in\Gamma,\quad \int_{\gamma x}^{\gamma y}\widetilde{F}=\int_x^y\widetilde{F}\qquad\text{and}\qquad \int_y^x\widetilde{F}=\int_x^y\widetilde{F}\circ\iota\;. \tag{13}$$

For every hyperbolic element γ in Γ, the *period of* γ for the potential F is

$$\mathrm{Per}_F(\gamma)=\int_x^{\gamma x}\widetilde{F}$$

for any point x on the translation axis of γ in $\widetilde{M}$. Note that, for all $n\in\mathbb{N}-\{0\}$ and $\alpha\in\Gamma$, we have

$$\mathrm{Per}_F(\alpha\gamma\alpha^{-1})=\mathrm{Per}_F(\gamma),\quad \mathrm{Per}_F(\gamma^n)=n\ \mathrm{Per}_F(\gamma)\quad\text{and}\quad \mathrm{Per}_F(\gamma^{-1})=\mathrm{Per}_{F\circ\iota}(\gamma)\;. \tag{14}$$

Let $\widetilde{F}^*:T^1\widetilde{M}\to\mathbb{R}$ be another Hölder-continuous Γ-invariant map. Say that $\widetilde{F}^*$ is *cohomologous* to $\widetilde{F}$ (see for instance [Livs]) if there exists a Hölder-continuous Γ-invariant map $\widetilde{G}:T^1\widetilde{M}\to\mathbb{R}$, differentiable along every flow line, such that

$$\widetilde{F}^*(v)-\widetilde{F}(v)=\frac{d}{dt}_{|t=0}\widetilde{G}(\phi_t v)\;.$$

We will say that $\widetilde{F}^*$ is *cohomologous to* $\widetilde{F}$ *via* $\widetilde{G}$ when we want to emphasise $\widetilde{G}$. Two cohomologous potentials have the same periods: if $\widetilde{F}^*$ is cohomologous to $\widetilde{F}$, then, denoting by F^* the map induced on T^1M by $\widetilde{F}^*$, for every hyperbolic element $\gamma\in\Gamma$, we have

$$\mathrm{Per}_F(\gamma)=\mathrm{Per}_{F^*}(\gamma)\;.$$

Remark 3.1 *Let $\widetilde{F},\widetilde{F}^*:T^1\widetilde{M}\to\mathbb{R}$ be Hölder-continuous Γ-invariant maps such that* $\mathrm{Per}_F(\gamma)=\mathrm{Per}_{F^*}(\gamma)$ *for every hyperbolic element $\gamma\in\Gamma$. Then $\widetilde{F}^*$ is cohomologous to $\widetilde{F}$ in restriction to $\widetilde{\Omega}\Gamma$.*

Proof. Since the diagonal action of Γ on $\Lambda\Gamma\times\Lambda\Gamma$ is topologically transitive, the action of the geodesic flow on the (topological) nonwandering set $\Omega\Gamma$ is topologically transitive. The validity of Anosov's closing lemma (see Lemma 2.6) does not require any compactness assumption on M. The proof of Livšic's theorem in [KaH, Theo. 19.2.4] then extends. $\square$

Note that if $\widetilde{F}^*$ and $\widetilde{F}$ are cohomologous via the map $\widetilde{G}$, then $\widetilde{F}^*\circ\iota$ and $\widetilde{F}\circ\iota$ are cohomologous via the map $-\widetilde{G}\circ\iota$ (beware the sign). We will say that $\widetilde{F}$ (or F) is *reversible* if $\widetilde{F}$ is cohomologous to $\widetilde{F}\circ\iota$.

We end this subsection by the following technical lemma on potential functions (see also [Cou2, Lem. 3] with the multiplicative approach).

Lemma 3.2 *For every $r_0 > 0$, there exist two constants $c_1 > 0$ and $c_2 \in\,]0,1]$ (depending, besides r_0 for c_1, only on the Hölder constants of $\widetilde{F}$ and the bounds on the sectional curvature of $\widetilde{M}$) such that for all x, y, z in $\widetilde{M}$, we have*

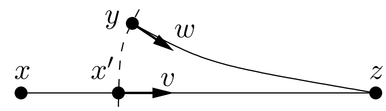
$$\Big|\int_x^z \widetilde{F} \;-\int_y^z \widetilde{F}\;\Big| \leq c_1\, e^{d(x,\,y)} + d(x,y) \max_{\pi^{-1}(B(x,\,d(x,y)))} |\widetilde{F}|\;,$$

and if furthermore $d(x,y) \leq r_0$, then

$$\Big|\int_x^z \widetilde{F} \;-\int_y^z \widetilde{F}\;\Big| \leq c_1\, d(x,y)^{c_2} + d(x,y) \max_{\pi^{-1}(B(x,d(x,y)))} |\widetilde{F}|\;.$$

Proof. By symmetry, we may assume that $d(x,z) \geq d(y,z)$. The result is true if $y = z$, hence we assume that $y \neq z$.

Let x' be the point on $[x,z]$ at distance $d(y,z)$ from z. Let v (respectively w) be the unit tangent vector at x' (respectively y) pointing towards z.



The closest point p of y on $[x,z]$ lies in $[x',z]$ by convexity. Hence $d(x,x') \leq d(x,p) \leq d(x,y)$, since closest point maps do not increase distances. Since the distance function from a given point to a point varying on a geodesic line is convex, we have $d(x',y) \leq d(x,y)$.

Let $c > 0$ and $\alpha \in\,]0,1]$ be the Hölder constants of $\widetilde{F}$ for the distance $d' = d'_{T^1\widetilde{M}}$ on $T^1\widetilde{M}$ defined in Equation (7), that is $|\widetilde{F}(u) - \widetilde{F}(u')| \leq c\, d'(u,u')^\alpha$ for all u, u' in $T^1\widetilde{M}$ with $d'(u,u') \leq 1$. By Corollary 2.5 (i), there exists a universal constant $c' > 0$ (for instance $c' = \frac{e^2}{2}$) such that for every $t \in [0, d(y,z)]$, we have $d'(\phi_t v, \phi_t w) \leq c' e^{d(x',\,y)-t}$.

Let $t \in [0, d(y,z)]$. If $d'(\phi_t v, \phi_t w) \leq 1$, then

$$|\widetilde{F}(\phi_t v) - \widetilde{F}(\phi_t w)| \leq c\, d'(\phi_t v, \phi_t w)^\alpha \leq c\,\big(c'\, e^{d(x',\,y)-t}\big)^\alpha\;.$$

If $d'(\phi_t v, \phi_t w) \geq 1$, then by the remark at the end of Subsection 2.1, we have

$$|\widetilde{F}(\phi_t v) - \widetilde{F}(\phi_t w)| \leq 3\,c\, d'(\phi_t v, \phi_t w) \leq 3\,c\,c'\, e^{d(x',\,y)-t}\;.$$

Hence,

$$\begin{aligned}
\Big|\int_x^z \widetilde{F} - \int_y^z \widetilde{F}\;\Big| &= \Big|\int_{-d(x,\,x')}^{d(y,\,z)} \widetilde{F}(\phi_t v)\, dt - \int_0^{d(y,\,z)} \widetilde{F}(\phi_t w)\, dt\;\Big| \\
&\leq \int_{-d(x,\,x')}^{0} |\widetilde{F}(\phi_t v)|\, dt + \int_0^{d(y,\,z)} |\widetilde{F}(\phi_t v) - \widetilde{F}(\phi_t w)|\, dt \\
&\leq d(x,x') \max_{\pi^{-1}(B(x,\,d(x,x')))} |\widetilde{F}| + \int_0^{+\infty} c\,\big(c'\, e^{d(x',\,y)-t}\big)^\alpha + 3\,c\,c'\, e^{d(x',\,y)-t}\, dt \\
&\leq d(x,y) \max_{\pi^{-1}(B(x,\,d(x,y)))} |\widetilde{F}| + \Big(3\,c\,c' + \frac{c(c')^\alpha}{\alpha}\Big)\, e^{d(x,\,y)}\;.
\end{aligned}$$

The first result follows. The second may be proved similarly, using Corollary 2.5 (ii). □

Remarks. (1) When $x, y, z \in \mathscr{C}\Lambda\Gamma$, we may replace, in each assertion of the above lemma, $\max_{\pi^{-1}(B(x,\,d(x,y)))} |\widetilde{F}|$ by $\max_{\pi^{-1}(B(x,\,d(x,y))\,\cap\,\mathscr{C}\Lambda\Gamma)} |\widetilde{F}|$.

(2) Using the equality $\int_x^{x'} \widetilde{F} - \int_y^{y'} \widetilde{F} = \big(\int_x^{x'} \widetilde{F} - \int_y^{x'} \widetilde{F}\big) + \big(\int_{x'}^{y} \widetilde{F}\circ\iota - \int_{y'}^{y} \widetilde{F}\circ\iota\big)$ and the fact that ι is Lipschitz with constants depending only on the bounds the sectional curvature of $\widetilde{M}$, we have a similar control on $\big|\int_x^{x'} \widetilde{F} - \int_y^{y'} \widetilde{F}\;\big|$ for all x, x', y, y' in $\widetilde{M}$, in terms of $\max\{d(x,y), d(x',y')\}$.

3.2 The Poincaré series and the critical exponent of (Γ, F)

Let us fix $x, y \in \widetilde{M}$. The *Poincaré series* of (Γ, F) is the map $Q_{\Gamma,\,F} = Q_{\Gamma,\,F,\,x,\,y} : \mathbb{R} \to [0, +\infty]$ defined by

$$Q_{\Gamma,\,F,\,x,\,y}(s) = \sum_{\gamma \in \Gamma} \; e^{\int_x^{\gamma y} (\widetilde{F} - s)} \; .$$

The *critical exponent* of (Γ, F) is the element $\delta_{\Gamma,\,F}$ in $[-\infty, +\infty]$ defined by

$$\delta_{\Gamma,\,F} = \limsup_{n \to +\infty} \; \frac{1}{n} \log \sum_{\gamma \in \Gamma,\; n-1 < d(x, \gamma y) \leq n} \; e^{\int_x^{\gamma y} \widetilde{F}} \; .$$

When $F = 0$, the Poincaré series $Q_\Gamma = Q_{\Gamma,\,x,\,y}$ of $(\Gamma, 0)$ is the usual *Poincaré series* of Γ, and the critical exponent $\delta_{\Gamma,\,0}$ is the *critical exponent* δ_Γ of Γ (which belongs to $]0, +\infty[$ since Γ is nonelementary and $\widetilde{M}$ has pinched negative curvature, see for instance [Rob1]).

We will prove in the following Lemma 3.3 (v) that $\delta_{\Gamma,\,F} > -\infty$. If $\delta_{\Gamma,\,F} < +\infty$, we say that (Γ, F) is *of divergence type* if the series $Q_{\Gamma,\,F}(\delta_{\Gamma,\,F})$ diverges, and *of convergence type* otherwise.

If $\delta_{\Gamma,\,F} < +\infty$, the *normalised potential* is the Hölder-continuous map $\widetilde{F} - \delta_{\Gamma,\,F}$ on $T^1\widetilde{M}$ (or its induced map $F - \delta_{\Gamma,\,F}$ on T^1M). Some references use the normalised potential with the opposite sign.

Since $\widetilde{F}$ is Hölder-continuous (see Remark (2) at the end of the previous Subsection 3.1), the critical exponent $\delta_{\Gamma,\,F}$ of (Γ, F) does not depend on x, y, and satisfies the following elementary properties.

Lemma 3.3 *(i) The Poincaré series of (Γ, F) converges if $s > \delta_{\Gamma,\,F}$ and diverges if $s < \delta_{\Gamma,\,F}$. The Poincaré series of (Γ, F) diverges at $\delta_{\Gamma,\,F}$ if and only if the Poincaré series of $(\Gamma, F + \kappa)$ diverges at $\delta_{\Gamma,\,F+\kappa}$, for every $\kappa \in \mathbb{R}$, and in particular*

$$\forall\, \kappa \in \mathbb{R}, \quad \delta_{\Gamma,\,F+\kappa} = \delta_{\Gamma,\,F} + \kappa \; . \tag{15}$$

(ii) We have

$$\forall\, s \in \mathbb{R},\; Q_{\Gamma,\,F\circ\iota,\,x,\,y}(s) = Q_{\Gamma,\,F,\,y,\,x}(s) \quad \text{and} \quad \delta_{\Gamma,\,F\circ\iota} = \delta_{\Gamma,\,F} \; . \tag{16}$$

(iii) If Γ' is a nonelementary subgroup of Γ, denoting by $F' : \Gamma' \backslash T^1M \to \mathbb{R}$ the map induced by $\widetilde{F}$, we have

$$\delta_{\Gamma',\,F'} \leq \delta_{\Gamma,\,F} \; .$$

(iv) We have the upper and lower bounds

$$\delta_\Gamma + \inf_{\pi^{-1}(\mathscr{C}\Lambda\Gamma)} \widetilde{F} \;\leq\; \delta_{\Gamma,\,F} \;\leq\; \delta_\Gamma + \sup_{\pi^{-1}(\mathscr{C}\Lambda\Gamma)} \widetilde{F} \; .$$

(v) We have $\delta_{\Gamma,\,F} > -\infty$.
(vi) The map $F \mapsto \delta_{\Gamma,\,F}$ is convex, subadditive and 1-Lipschitz for the uniform norm on the vector space of real continuous maps on $\pi^{-1}(\mathscr{C}\Lambda\Gamma)$, that is, if $\widetilde{F^*} : T^1\widetilde{M} \to \mathbb{R}$ is another Hölder-continuous Γ-invariant map, inducing $F^* : \Gamma \backslash T^1M \to \mathbb{R}$, if $\delta_{\Gamma,\,F}, \delta_{\Gamma,\,F^*} < +\infty$, then

$$|\; \delta_{\Gamma,\,F^*} - \delta_{\Gamma,\,F} \;| \leq \sup_{v \in \pi^{-1}(\mathscr{C}\Lambda\Gamma)} |\; \widetilde{F^*}(v) - \widetilde{F}(v) \;| \; ,$$

$$\delta_{\Gamma,\,F+F^*} \leq \delta_{\Gamma,\,F} + \delta_{\Gamma,\,F^*}\ ,$$

and, for every $t \in [0,1]$,

$$\delta_{\Gamma,\,tF+(1-t)F^*} \leq t\,\delta_{\Gamma,\,F} + (1-t)\,\delta_{\Gamma,\,F^*}\ .$$

(vii) We have

$$\delta_{\Gamma,\,F} = \limsup_{n\to+\infty} \frac{1}{n} \log \sum_{\gamma\in\Gamma,\ d(x,\,\gamma y)\leq n} e^{\int_x^{\gamma y} \widetilde{F}}\ . \tag{17}$$

We will prove in Subsection 4.2 that the upper limit in Equation (17) is in fact a limit.

An interesting problem is to study whether or not the critical exponent $\delta_{\Gamma,\,F}$ of (Γ, F) is equal to the upper bound of the critical exponents $\delta_{\Gamma_0,\,F_0}$ where Γ_0 ranges over the convex-cocompact subgroups of Γ and $F_0 : \Gamma_0\backslash T^1M \to \mathbb{R}$ is the map induced by $\widetilde{F}$. Replacing subgroups by subsemigroups, we answer this positively in Subsection 4.4.

Proof. The verifications of (i), (ii) and (iii) are elementary. For instance, Equation (16) follows from both parts of Equation (13) and the change of variable $\gamma \mapsto \gamma^{-1}$ in the Poincaré series.

To prove Assertion (iv), note that if x is a point in the convex hull of the limit set $\mathscr{C}\Lambda\Gamma$, then, for every $\gamma \in \Gamma$, the geodesic segment between x and γx is contained in $\mathscr{C}\Lambda\Gamma$. Hence

$$d(x,\gamma x)\Big(\inf_{\pi^{-1}(\mathscr{C}\Lambda\Gamma)} \widetilde{F} - s\Big) \leq \int_x^{\gamma x} (\widetilde{F} - s) \leq d(x,\gamma x)\Big(\sup_{\pi^{-1}(\mathscr{C}\Lambda\Gamma)} \widetilde{F} - s\Big)\ .$$

This proves Assertion (iv) for instance by taking the exponential, summing over $\gamma \in \Gamma$ and using the first assertion of (i).

To prove Assertion (v), let Γ' be a nonelementary convex-cocompact subgroup of Γ (for instance a Schottky subgroup of Γ). Denote by $F' : \Gamma'\backslash T^1M \to \mathbb{R}$ the map induced by $\widetilde{F}$. Since $|\widetilde{F}|$ is Γ-invariant and bounded on compact subsets of $\widetilde{M}$, by Assertion (iv), we have $\delta_{\Gamma',\,F'} > -\infty$. Assertion (v) then follows from Assertion (iii).

To prove Assertion (vi), let $c = \sup_{\pi^{-1}(\mathscr{C}\Lambda\Gamma)} |\widetilde{F} - \widetilde{F^*}|$ and $x \in \mathscr{C}\Lambda\Gamma$. We have

$$\int_x^{\gamma x} (\widetilde{F} - (s+c)) \leq \int_x^{\gamma x} (\widetilde{F^*} - s) \leq \int_x^{\gamma x} (\widetilde{F} - (s-c))\ ,$$

and the first claim follows. To prove the two remaining claims of (vi), for all $s > \delta_{\Gamma,\,F}$ and $s^* > \delta_{\Gamma,\,F^*}$, we have

$$Q_{\Gamma,\,F,\,x,\,y}(s)\,Q_{\Gamma,\,F^*,\,x,\,y}(s^*) \geq Q_{\Gamma,\,F+F^*,\,x,\,y}(s+s^*)$$

and, by the convexity of the exponential,

$$Q_{\Gamma,\,tF+(1-t)F^*,\,x,\,y}(ts + (1-t)s^*) \leq tQ_{\Gamma,\,F,\,x,\,y}(s) + (1-t)Q_{\Gamma,\,F^*,\,x,\,y}(s^*)\ .$$

Hence $s + s^* \geq \delta_{\Gamma,\,F+F^*}$ and $\delta_{\Gamma,\,tF+(1-t)F^*} \leq ts + (1-t)s^*$ by the first assertion of (i). The result follows by letting s and s^* converge respectively to $\delta_{\Gamma,\,F}$ and $\delta_{\Gamma,\,F^*}$.

Finally, to prove Assertion (vii), for every $s \in \mathbb{R}$, let

$$a_{n,\,s} = \sum_{\gamma\in\Gamma,\ n-1<d(x,\gamma y)\leq n} e^{\int_x^{\gamma y}(\widetilde{F}-s)} \in [0,+\infty[\ ,$$

$\overline{a}_{n,\,s} = \sum_{k=0}^{n} a_{k,\,s}$, $\delta_s = \limsup_{n\to+\infty} \frac{1}{n}\log a_{n,\,s}$ and $\overline{\delta}_s = \limsup_{n\to+\infty} \frac{1}{n}\log \overline{a}_{n,\,s}$. Since $\overline{a}_{n,\,s} \geq a_{n,\,s}$, we have $\overline{\delta}_s \geq \delta_s = \delta_{\Gamma,\,F} - s$. In particular $\overline{\delta}_0 \geq \delta_{\Gamma,\,F}$ and if $\delta_{\Gamma,\,F} = +\infty$, then Assertion (vii) is true.

Assume now that $\delta_{\Gamma,\,F} < +\infty$. Since $\delta_{\Gamma,\,F} > -\infty$ by Assertion (v), let us fix $s < \delta_{\Gamma,\,F}$, so that $\delta_s > 0$. Let ρ_s (respectively $\overline{\rho}_s$) be the radius of convergence of the power series $\sum a_{n,\,s}\, z^n$ (respectively $\sum \overline{a}_{n,\,s}\, z^n$). Since these power series have nonnegative coefficients, we have $\delta_s = \log\frac{1}{\rho_s}$ and $\overline{\delta}_s = \log\frac{1}{\overline{\rho}_s}$. For every $z \in\,]0, \rho_s[$, the series $\sum a_{n,\,s}\, z^n$ converges, and $z < 1$ since $\rho_s = e^{-\delta_s} < 1$. Hence, by Fubini's theorem for series with nonnegative coefficients, the series

$$\sum_{n\in\mathbb{N}} \overline{a}_{n,\,s} z^n = \sum_{n\in\mathbb{N}}\sum_{k=0}^{n} a_{k,\,s} z^n = \sum_{k\in\mathbb{N}}\sum_{n=k}^{+\infty} a_{k,\,s} z^n = \frac{1}{1-z}\sum_{k\in\mathbb{N}} a_{k,\,s} z^k$$

converges. Since $\overline{a}_{n,\,s} \geq 0$ for every $n \in \mathbb{N}$, we have $\overline{\rho}_s \geq z$, hence $\overline{\rho}_s \geq \rho_s$ and $\overline{\delta}_s \leq \delta_s = \delta_{\Gamma,\,F} - s$. Since $\overline{\delta}_s \geq \overline{\delta}_0 - s$, we therefore have $\overline{\delta}_0 \leq \overline{\delta}_s + s \leq \delta_{\Gamma,\,F}$. As we had proved that $\overline{\delta}_0 \geq \delta_{\Gamma,\,F}$, we have $\overline{\delta}_0 = \delta_{\Gamma,\,F}$, which gives Equation (17). $\square$

Here are a few immediate consequences. The convergence or divergence of the Poincaré series $Q_{\Gamma,\,F}(s)$ is independent of the points x and y. The Poincaré series $Q_{\Gamma,\,F\circ\iota}(s)$ converges if and only if $Q_{\Gamma,\,F}(s)$ does. The critical exponent $\delta_{\Gamma,\,F}$ of (Γ, F) is positive if $\inf_{T^1\widetilde{M}} \widetilde{F} > -\delta_\Gamma$. For instance, $\delta_{\Gamma,\,F} > 0$ if F is a small pertubation of zero (more precisely if $\|F\|_\infty = \sup_{T^1\widetilde{M}} |\widetilde{F}| < \delta_\Gamma$), or if $F \geq 0$, since Γ is nonelementary and therefore $\delta_\Gamma > 0$). Recall that $\delta_\Gamma < +\infty$ since $\widetilde{M}$ has pinched curvature. Hence if F is bounded from above, then $\delta_{\Gamma,\,F} < +\infty$.

When $\delta_{\Gamma,\,F} < +\infty$, the additive convention for the Poincaré series and the fact that its interesting behaviour occurs at $s = \delta_{\Gamma,\,F}$ suggest it is meaningful to consider the normalised potential $F - \delta_{\Gamma,\,F}$. By Equation (15), the critical exponent of the normalised potential is 0. Moreover, adding a large enough constant to F does not change the normalised potential and allows the critical exponent of (Γ, F) to be positive (this will be a standing assumption in most of the results of the chapters 9 and 10).

The relationship between $\delta_{\Gamma,\,F}$ and $\delta_{\Gamma,\,sF}$ for $s > 0$ seems unclear in general.

Remark. Let $\widetilde{F}' : T^1\widetilde{M} \to \mathbb{R}$ be another Hölder-continuous Γ-invariant map, which is cohomologous to $\widetilde{F}$ via $\widetilde{G}$. Let F' be the map induced on $T^1M = \gamma\backslash T^1\widetilde{M}$ by $\widetilde{F}'$. Note that for all $x, y \in \widetilde{M}$ and $\gamma \in \Gamma$, by Γ-invariance of $\widetilde{G}$,

$$\Big|\int_x^{\gamma y} \widetilde{F} - \int_x^{\gamma y} \widetilde{F}'\,\Big| = \big|\widetilde{G}(\phi_{d(x,\gamma y)}v) - \widetilde{G}(v)\big| \leq 2 \sup_{w\,\in\, T^1_x\widetilde{M}\,\cup\, T^1_y\widetilde{M}} |\widetilde{G}(w)|\ ,$$

where v is a unit tangent vector such that $\pi(v) = x$ and $\pi(\phi_{d(x,\gamma y)}v) = \gamma y$. Hence F and F' have the same critical exponent:

$$\delta_{\Gamma,\,F'} = \delta_{\Gamma,\,F}\ .$$

Furthermore, (Γ, F') is of divergence type if and only if (Γ, F) is of divergence type.

3.3 The Gibbs cocycle of (Γ, F)

By the Hölder-continuity of F and the properties of asymptotic geodesic rays in $\widetilde{M}$, there exists a well-defined map $C_F : \partial_\infty \widetilde{M} \times \widetilde{M} \times \widetilde{M} \to \mathbb{R}$, called the *Gibbs cocycle* for the potential F, defined by

$$(\xi, x, y) \mapsto C_{F,\xi}(x, y) = \lim_{t\to+\infty} \int_y^{\xi_t} \widetilde{F} - \int_x^{\xi_t} \widetilde{F}$$

for $t \mapsto \xi_t$ any geodesic ray ending at ξ. Do note the apparent order reversal of x and y in this formula, which allows us to have the required sign in front of the Busemann cocycle in Equation (20). We will compare our cocycle with the one introduced by Hamenstädt in [Ham2] at the end of Subsection 3.5.

If x belongs to the geodesic ray from y to ξ, then

$$C_{F,\xi}(x, y) = \int_y^x \widetilde{F} \ . \tag{18}$$

In particular, for every $w \in T^1\widetilde{M}$, for all x and y on the image of the geodesic line defined by w, with w_-, x, y, w_+ in this order, we have

$$C_{F\circ\iota,\, w_-}(x, y) = \int_y^x \widetilde{F} \circ \iota = \int_x^y \widetilde{F} = C_{F,\, w_+}(y, x) = -C_{F,\, w_+}(x, y) \ . \tag{19}$$

Note that when $F = -1$, the Gibbs cocycle equals the Busemann cocycle (defined at the end of Subsection 2.2)

$$\beta_\xi(x, y) = \lim_{t\to+\infty} d(x, \xi_t) - d(y, \xi_t)$$

with $t \mapsto \xi_t$ as above. Hence, for every $s \in \mathbb{R}$,

$$C_{F-s,\,\xi}(x, y) = C_{F,\,\xi}(x, y) + s\ \beta_\xi(x, y) \ . \tag{20}$$

The Gibbs cocycle satisfies the following cocycle property: for all ξ in $\partial_\infty \widetilde{M}$ and x, y, z in $\widetilde{M}$,

$$C_{F,\xi}(x, z) = C_{F,\xi}(x, y) + C_{F,\xi}(y, z) \ \text{ and } \ C_{F,\xi}(x, y) = -\, C_{F,\xi}(y, x) \ , \tag{21}$$

and the following invariance property: for all γ in Γ, ξ in $\partial_\infty \widetilde{M}$ and x, y in $\widetilde{M}$,

$$C_{F,\gamma\xi}(\gamma x, \gamma y) = C_{F,\xi}(x, y) \ . \tag{22}$$

Remarks. (1) If $\widetilde{F}^* : T^1\widetilde{M} \to \mathbb{R}$ is a Hölder-continuous Γ-invariant map, which is cohomologous to $\widetilde{F}$, then the cocycle $C_{F^*} - C_F$ is a coboundary. Indeed, and more precisely, for all $x, y \in \widetilde{M}$, for every $\xi \in \partial_\infty \widetilde{M}$, if $\widetilde{F}^*$ is cohomologous to $\widetilde{F}$ via the map $\widetilde{G} : T^1\widetilde{M} \to \mathbb{R}$, if $v_{z\xi}$ is the tangent vector at the origin of the geodesic ray from a point $z \in \widetilde{M}$ to ξ, then

$$C_{F^*,\xi}(x, y) - C_{F,\xi}(x, y) = \widetilde{G}(v_{x\xi}) - \widetilde{G}(v_{y\xi}) \ . \tag{23}$$

In particular, $C_{F^*} - C_F$ is bounded on $\partial_\infty \widetilde{M} \times \widetilde{M} \times \widetilde{M}$ if $\widetilde{G}$ is bounded.

(2) Note that the Gibbs cocycle C_F entirely determines the potential function $\widetilde{F}$, since for every $v \in T^1\widetilde{M}$, we have

$$\widetilde{F}(v) = \lim_{t\to 0^+} \frac{1}{t}\, C_{F,\, v_+}(\pi(v), \pi(\phi_t v)) \ .$$

(3) For every $x_0 \in \widetilde{M}$, the map $(\xi, \gamma) \mapsto C_{F,\xi}(x_0, \gamma x_0)$ from $\Lambda\Gamma \times \Gamma$ to $\mathbb{R}$ determines, up to a cohomologous potential, the restriction of $\widetilde{F}$ to $\widetilde{\Omega}\Gamma$.

Indeed, let y_0 be the closest point to x_0 on the translation axis of a hyperbolic element $\gamma \in \Gamma$. Let γ_+ be the attractive fixed point of γ. By the equations (21), (22) and (18), we have

$$C_{F,\gamma^+}(x_0, \gamma x_0) = C_{F,\gamma^+}(x_0, y_0) + C_{F,\gamma^+}(y_0, \gamma y_0) + C_{F,\gamma^+}(\gamma y_0, \gamma x_0) = \int_{y_0}^{\gamma y_0} \widetilde{F}\,.$$

Hence

$$\mathrm{Per}_F(\gamma) = C_{F,\gamma^+}(x_0, \gamma x_0)\;,$$

The result now follows from Remark 3.1.

Lemma 3.4 *For every $r_0 > 0$, there exist $c_1, c_2, c_3, c_4 > 0$ with $c_2, c_4 \leq 1$ (depending only on r_0, the Hölder constants of $\widetilde{F}$ and the bounds on the sectional curvature of $\widetilde{M}$) such that the following assertions hold.*

(1) For all $x, y \in \widetilde{M}$ and $\xi \in \partial_\infty \widetilde{M}$,

$$|\, C_{F,\xi}(x,y)\,| \leq\; c_1\, e^{d(x,\,y)} \;+\; d(x,y) \max_{\pi^{-1}(B(x,\,d(x,y)))} |\widetilde{F}|\;,$$

and if furthermore $d(x, y) \leq r_0$, then

$$|\, C_{F,\xi}(x,y)\,| \leq c_1\, d(x,y)^{c_2} + d(x,y) \max_{\pi^{-1}(B(x,\,d(x,y)))} |\widetilde{F}|\;.$$

(2) For every $r \in [0, r_0]$, for all x, y' in $\widetilde{M}$, for every ξ in the shadow $\mathscr{O}_x B(y', r)$ of the ball $B(y', r)$ seen from x, we have

$$\Big|\; C_{F,\xi}(x,y') + \int_x^{y'} \widetilde{F}\;\Big| \leq c_3\, r^{c_4} + 2r \max_{\pi^{-1}(B(y',\,r))} |\widetilde{F}|\;.$$

Proof. (1) This follows from Lemma 3.2 by letting z tend to ξ along a geodesic ray.

(2) Let $t \mapsto \xi_t$ be the geodesic ray starting from x and ending at ξ, and let p be the closest point on it to y', which satisfies $d(y', p) \leq r$. Since ι is Lipschitz for d'_{T^1M} (see the paragraph before Lemma 2.2) with constant depending only on the bounds of the sectional curvature, the assertion (2) follows from the second assertion of Lemma 3.2, using

$$\int_x^{\xi_t} \widetilde{F} - \int_{y'}^{\xi_t} \widetilde{F} - \int_x^{y'} \widetilde{F} = \Big(\int_p^x \widetilde{F}\circ\iota - \int_{y'}^x \widetilde{F}\circ\iota\Big) + \Big(\int_p^{\xi_t} \widetilde{F} - \int_{y'}^{\xi_t} \widetilde{F}\Big)\,. \quad \square$$

Remark. By Remark (1) at the end of Subsection 3.1, when $x, y \in \mathscr{C}\Lambda\Gamma$ and $\xi \in \Lambda\Gamma$, we may replace $\max_{\pi^{-1}(B(x,\,d(x,y)))} |\widetilde{F}|$ by $\max_{\pi^{-1}(B(x,\,d(x,y))\cap\mathscr{C}\Lambda\Gamma)} |\widetilde{F}|$ in Assertion (1) of the above lemma.

Proposition 3.5 *Assume that $\widetilde{F}$ is bounded. Then the map $C_F : \partial_\infty \widetilde{M} \times \widetilde{M} \times \widetilde{M} \to \mathbb{R}$ is locally Hölder-continuous. Furthermore, the maps $\xi \mapsto C_{F,\xi}(x,y)$ for all $x, y \in \widetilde{M}$ and $(x,y) \mapsto C_{F,\xi}(x,y)$ for every $\xi \in \partial_\infty \widetilde{M}$ are Hölder-continuous.*

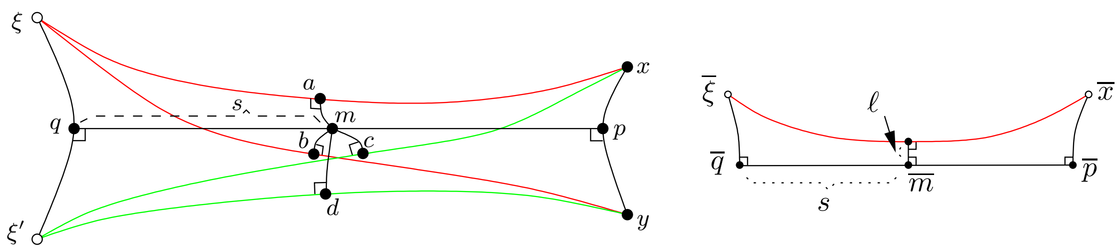
In particular, the Busemann cocycle (obtained by taking $F = -1$) is locally Hölder-continuous.

If we assume only that $\widetilde{F}$ is bounded on $\mathscr{C}\Lambda\Gamma$, then the same proof gives that the restriction of C_F to $\Lambda\Gamma \times \widetilde{M} \times \widetilde{M} \to \mathbb{R}$ is locally Hölder-continuous.

Proof. Let $x, y, x', y' \in \widetilde{M}$ and $\xi, \xi' \in \partial_\infty \widetilde{M}$.

Let us first prove that there exist constants $c > 0$ and $\alpha \in \,]0, 1[$ (depending only on $d(x, y)$, on the Hölder constants of $\widetilde{F}$, on the upper bound of $|\widetilde{F}|$ and on the bounds on the sectional curvature of $\widetilde{M}$) such that if $d_x(\xi, \xi') \leq e^{-d(x,y)-2}$, then

$$|C_{F,\,\xi'}(x, y) - C_{F,\,\xi}(x, y)| \leq c\, d_x(\xi, \xi')^\alpha \,.$$



Let $[p, q]$ be the shortest arc between $[x, y]$ and $]\xi, \xi'[$, with $p \in [x, y]$. Let m be the midpoint of $[p, q]$ and $s = d(p, m) = d(m, q) = \frac{1}{2} d(p, q)$. Let a, b, c, d be the closest points to m on respectively $[x, \xi[$, $[x, \xi'[$, $[y, \xi[$, $[y, \xi'[$.

Since q is the closest point to p on $]\xi, \xi'[$, and by comparison, the distance from q to $[p, \xi[$ and to $[p, \xi'[$ is at most $\log(1 + \sqrt{2})$. Hence by the triangle inequality and the definition of the visual distance (see Equation (1)), we have

$$e^{-2s} \leq d_p(\xi, \xi') \leq e^{-2s+2\log(1+\sqrt{2})} \,.$$

By the properties of the visual distances (see Equation (2)), we have

$$e^{-d(x,\,y)} \leq e^{-d(x,\,p)} \leq \frac{d_x(\xi, \xi')}{d_p(\xi, \xi')} \leq e^{d(x,\,p)} \leq e^{d(x,\,y)} \,.$$

In particular,

$$s \geq -\frac{1}{2} \log d_p(\xi, \xi') \geq \frac{1}{2}(-\log d_x(\xi, \xi') - d(x, y)) \geq 1 \,.$$

By comparison, since the angle at q between p and ξ is $\frac{\pi}{2}$, and since, if $p \neq x$, the angle at p between q and x is at least $\frac{\pi}{2}$ (and exactly $\frac{\pi}{2}$ if $p \neq y$), the distance from m to a is at most the distance ℓ in the real hyperbolic plane $\mathbb{H}^2_\mathbb{R}$ between the midpoint $\overline{m}$ of a segment $[\overline{p}, \overline{q}]$ of length $2s$ to a geodesic line between $\overline{\xi}, \overline{x} \in \partial_\infty \mathbb{H}^2_\mathbb{R}$ where the angle at $\overline{q}$ between $\overline{p}$ and $\overline{\xi}$ and the angle at $\overline{p}$ between $\overline{q}$ and $\overline{x}$ are exactly $\frac{\pi}{2}$. By a well known formula in hyperbolic geometry (see for instance [Bea, page 157]), we have $\sinh \ell = \frac{1}{\sinh s}$. Note that for all $s \geq 1$, we have $\sinh s \geq \frac{e^s}{4}$. Hence

$$d(a, m) \leq \ell \leq \sinh \ell = \frac{1}{\sinh s} \leq 4\, e^{-s} \leq 4\, d_p(\xi, \xi')^{\frac{1}{2}} \leq 4\, e^{\frac{d(x,\,y)}{2}}\, d_x(\xi, \xi')^{\frac{1}{2}} \,.$$

Since the computations are unchanged by permuting x and y, as well as ξ and ξ', we have

$$\max\{d(a, m),\, d(b, m),\, d(c, m),\, d(d, m)\} \leq 4\, e^{\frac{d(x,\,y)}{2}}\, d_x(\xi, \xi')^{\frac{1}{2}} \,.$$

Hence by the triangle inequality,

$$\max\{d(a,b),\, d(b,c),\, d(c,d),\, d(d,a)\} \leq 8\, e^{\frac{d(x,\,y)}{2}}\, d_x(\xi,\xi')^{\frac{1}{2}} \leq 8\, e^{\frac{d(x,\,y)}{2}}\ . \tag{24}$$

Now, using respectively

- the cocycle property (21),
- Equation (18),
- the triangle inequality,
- Lemma 3.2 and Lemma 3.4 with $r_0 = 8\, e^{\frac{d(x,\,y)}{2}}$, for some constants $c_1 > 0$ and $c_2 \in\,]0,1[$ (depending only on r_0, the Hölder constants of $\widetilde{F}$, and the bounds on the sectional curvature of $\widetilde{M}$) and $\kappa = \sup_{\widetilde{M}} |\widetilde{F}|$,
- Equation (24),

we have

$$\begin{aligned}
&|C_{F,\,\xi'}(x,y) - C_{F,\,\xi}(x,y)|\\
&= |C_{F,\,\xi'}(x,c) + C_{F,\,\xi'}(c,d) + C_{F,\,\xi'}(d,y) - C_{F,\,\xi}(x,a) - C_{F,\,\xi}(a,b) - C_{F,\,\xi}(b,y)|\\
&= \Big| -\int_c^x \widetilde{F} + C_{F,\,\xi'}(c,d) + \int_d^y \widetilde{F} + \int_a^x \widetilde{F} - C_{F,\,\xi}(a,b) - \int_b^y \widetilde{F}\Big|\\
&\leq \Big|\int_a^x \widetilde{F} - \int_c^x \widetilde{F}\ \Big| + |C_{F,\,\xi'}(c,d)| + |C_{F,\,\xi}(a,b)| + \Big|\int_b^y \widetilde{F} - \int_d^y \widetilde{F}\ \Big|\\
&\leq c_1\, d(a,c)^{c_2} + \kappa\, d(a,c) + c_1\, d(c,d)^{c_2} + \kappa\, d(c,d)\\
&\qquad + c_1\, d(a,b)^{c_2} + \kappa\, d(a,b) + c_1\, d(b,d)^{c_2} + \kappa\, d(b,d)\\
&\leq 8^{1+c_2} \max\{c,\kappa\}\; e^{\frac{c_2\, d(x,y)}{2}}\, d_x(\xi,\xi')^{\frac{c_2}{2}}\ .
\end{aligned}$$

This proves the claim made at the beginning of this proof, as well as the second claim in the statement of Proposition 3.5.

Now, by Equation (21), we have

$$|C_{F,\,\xi'}(x',y') - C_{F,\,\xi}(x,y)| \leq |C_{F,\,\xi'}(x,y) - C_{F,\,\xi}(x,y)| + |C_{F,\,\xi'}(x,x')| + |C_{F,\,\xi'}(y,y')|\ .$$

Using the initial claim and again Lemma 3.4 (1), the result follows. □

3.4 The potential gap of (Γ, F)

For all $x \in \widetilde{M}$ and $(\xi,\eta) \in \partial^2_\infty \widetilde{M}$, define the F*-gap seen from* x between ξ and η by

$$D_{F,\,x}(\xi,\eta) = \exp \frac{1}{2}\Big(\lim_{t\to+\infty} \int_x^{\eta_t} \widetilde{F} - \int_{\xi_t}^{\eta_t} \widetilde{F} + \int_{\xi_t}^{x} \widetilde{F}\Big)\ ,$$

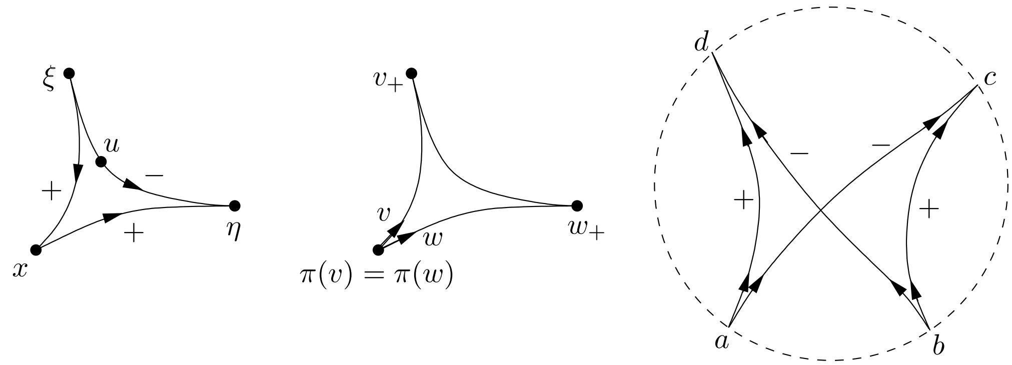
where $t \mapsto \xi_t$, $t \mapsto \eta_t$ are any geodesic rays ending at ξ and η respectively (see the left hand picture below). By the Hölder-continuity of F and the properties of asymptotic geodesic rays in $\widetilde{M}$, the limit does exist and is independent of the choices. We denote by

$$D_F : \widetilde{M} \times \partial^2_\infty \widetilde{M} \to \mathbb{R}$$

the map defined by $(x,\xi,\eta) \mapsto D_{F,\,x}(\xi,\eta)$, that we will call the *gap map* of the potential F. We will compare our map D_F with the analogous one introduced by Hamenstädt in [Ham2] at the end of Subsection 3.5. When F is the constant function with value -1, then

$D_{F,x}(\xi,\eta) = d_x(\xi,\eta)$, where d_x is the visual distance on $\partial_\infty \widetilde{M}$ seen from x (see Subsection 2.2). So that for every $s \in \mathbb{R}$, we have

$$D_{F-s,\,x}(\xi,\eta) = D_{F,\,x}(\xi,\eta)\; d_x(\xi,\eta)^s \;.$$



The gap map D_F satisfies the following elementary properties.

Lemma 3.6 *(1) For any point u on the geodesic line between ξ and η, we have*

$$D_{F,\,x}(\xi,\eta) = e^{-\frac{1}{2}(C_{F,\,\eta}(x,u)+C_{F\circ\iota,\,\xi}(x,u))}\;. \tag{25}$$

For every $\gamma \in \Gamma$,

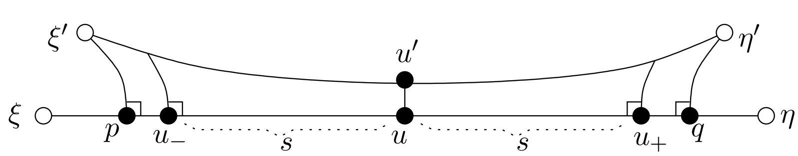
$$D_{F,\,\gamma x}(\gamma\xi,\gamma\eta) = D_{F,\,x}(\xi,\eta) \quad\text{and}\quad D_{F,\,x}(\eta,\xi) = D_{F\circ\iota,\,x}(\xi,\eta)\;. \tag{26}$$

(2) If $\widetilde{F}$ is bounded, the gap map $D_F : \widetilde{M} \times \partial^2_\infty \widetilde{M} \to \mathbb{R}$ is locally Hölder-continuous.

Proof. (1) This is immediate, using Equation (13) and the naturality under isometries of the construction.

(2) Let us first prove that there exists $c > 0$ such that for all $(\xi,\eta),(\xi',\eta') \in \partial^2_\infty \widetilde{M}$ and for every point u on the geodesic line $(\xi\eta)$ between ξ and η, if $d_u(\xi,\xi'), d_u(\eta,\eta') \le e^{-2}$, then there exists u' on the geodesic line $(\xi'\eta')$ such that

$$d(u,u') \le c\; \max\{d_u(\xi,\xi'), d_u(\eta,\eta')\}\;.$$



Define

$$s = \min\{-\log d_u(\xi,\xi'), -\log d_u(\eta,\eta')\} - \log(1+\sqrt{2}) \ge 1\;.$$

Let p be the closest point to ξ' on the geodesic ray $[u,\xi[$, which by comparison is at distance at most $\log(1+\sqrt{2})$ from the geodesic line $(\xi\xi')$. By the triangle inequality and the definition of the visual distance (see Equation (1)), we have $d(u,p) \ge -\log d_u(\xi,\xi') - \log(1+\sqrt{2})$. Similarly, the closest point q to η' on $[u,\eta[$ satisfies $d(u,q) \ge -\log d_u(\eta,\eta') - \log(1+\sqrt{2})$. By convexity, the point u_- at distance s from u on $[u,\xi[$ and the point u_+ at distance s from u on $[u,\eta[$ are the orthogonal projections of points on the extended

geodesic line $[\xi',\eta']$. By comparison (as in the proof of Proposition 3.5 using the formulae of [Bea, page 157]), the point u is at distance at most $4\,e^{-s}$ from the geodesic line $(\xi'\eta')$. The initial claim follows.

Now, for all $x,x'\in\widetilde{M}$ and $(\xi,\eta),(\xi',\eta')\in\partial^2_\infty\widetilde{M}$, fix u on the geodesic line $(\xi\eta)$, and let u' be the point defined above. By Equation (25), we have

$$|D_{F,\,x}(\xi,\eta)-D_{F,\,x'}(\xi',\eta')|=|e^{-\frac{1}{2}(C_{F,\,\eta}(x,u)+C_{F\circ\iota,\,\xi}(x,u))}-e^{-\frac{1}{2}(C_{F,\,\eta'}(x',u')+C_{F\circ\iota,\,\xi'}(x,u'))}|\;.$$

Assertion (2) of Lemma 3.6 then follows from Proposition 3.5. $\Box$

Remarks. (1) Let $\widetilde{F}':T^1\widetilde{M}\to\mathbb{R}$ be a Hölder-continuous Γ-invariant map, which is cohomologous to $\widetilde{F}$ via the map $\widetilde{G}:T^1\widetilde{M}\to\mathbb{R}$. Then for every $x\in\widetilde{M}$, for all $\xi,\eta\in\partial_\infty\widetilde{M}$, it follows from the equations (25) and (23), and from the fact that the potentials $\widetilde{F}'\circ\iota$ and $\widetilde{F}\circ\iota$ are cohomologous via the map $-\widetilde{G}\circ\iota$, that

$$D_{F',\,x}(\xi,\eta)=D_{F,\,x}(\xi,\eta)\;e^{\frac{1}{2}(\widetilde{G}(v_{x\xi})-\widetilde{G}\circ\iota(v_{x\eta}))}\;. \tag{27}$$

In particular, if F and $F\circ\iota$ are cohomologous, then by Equation (26), the ratio $\frac{D_{F,\,x}(\eta,\xi)}{D_{F,\,x}(\xi,\eta)}$ is uniformly bounded in (η,ξ), and also in (x,η,ξ) if the function $\widetilde{G}$ such that $\widetilde{F}$ and $\widetilde{F}\circ\iota$ are cohomologous via $\widetilde{G}$ is bounded.

(2) By Equation (25), by the cocycle property of C_F and by Lemma 3.4 (1), for all x,y in $\widetilde{M}$, there exists a constant $c_{x,\,y}>0$ such that, for all $\xi,\eta\in\partial_\infty\widetilde{M}$, we have

$$\frac{1}{c_{x,\,y}}\le\frac{D_{F,\,x}(\xi,\eta)}{D_{F,\,y}(\xi,\eta)}\le c_{x,\,y}\;.$$

(3) By the thin triangles property of CAT(-1) spaces, if p is the closest point to x on the geodesic line between ξ and η and $r=\log(1+\sqrt{2})$, we have $\eta,\xi\in\mathscr{O}_xB(p,r)$. Hence by Lemma 3.4 (2), if $\widetilde{F}$ is bounded, there exists a constant $c_5\ge1$ (depending only on $\|\widetilde{F}\|_\infty$ and on the Hölder constants of $\widetilde{F}$) such that, for all $x\in\widetilde{M}$ and $\xi,\eta\in\partial_\infty\widetilde{M}$, we have

$$\frac{1}{c_5}\le\frac{D_{F,\,x}(\xi,\eta)}{e^{\frac{1}{2}(\int_x^p\widetilde{F}+\int_x^p\widetilde{F}\circ\iota)}}\le c_5\;. \tag{28}$$

These inequalities are still satisfied, when $\widetilde{F}$ is only assumed to be bounded on $\pi^{-1}(\mathscr{C}\Lambda\Gamma)$, if $x\in\mathscr{C}\Lambda\Gamma$ and $\xi,\eta\in\Lambda\Gamma$.

(4) Let us prove in this remark that, under some assumptions on the potential, a minor modification of the gap map D_F yields a distance on $\Lambda\Gamma$ (see also [Sch1, Section 2.6]).

Lemma 3.7 *Let $x\in\mathscr{C}\Lambda\Gamma$. Assume that $\widetilde{F}$ is bounded on $\pi^{-1}(\mathscr{C}\Lambda\Gamma)$, and that either $\widetilde{F}$ is reversible or there exists a constant $c>0$ such that, for all $y,z\in\mathscr{C}\Lambda\Gamma$ such that $y\in[x,z]$, we have*

$$\int_x^z\widetilde{F}\le c+\int_x^y\widetilde{F}\;. \tag{29}$$

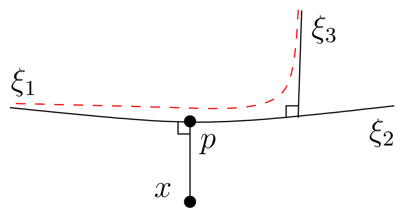
Then for every $\epsilon>0$ small enough, there exist a distance $d_{F,\,x,\,\epsilon}$ on $\Lambda\Gamma$ and $c_\epsilon>0$ such that for all $\xi,\eta\in\Lambda\Gamma$, we have

$$\frac{1}{c_\epsilon}\;D_{F,\,x}(\xi,\eta)^\epsilon\le d_{F,\,x,\,\epsilon}(\xi,\eta)\le c_\epsilon\;D_{F,\,x}(\xi,\eta)^\epsilon\;. \tag{30}$$

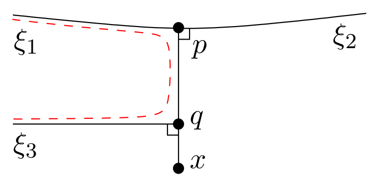
We will give, in the proof of Proposition 3.12, conditions on (Γ, F) implying that the hypothesis (29) is satisfied when F is a normalised potential. Note that the hypothesis (29) does hold when $\widetilde{F} \leq 0$ (for any Γ), as for instance when $\widetilde{F} = \widetilde{F}^{su}$ (see Chapter 7).

Proof. By the techniques of approximation by trees in CAT(-1)-spaces (see for instance [GdlH, §2.2]), there exists a universal constant $c' > 0$ such that for all $x \in \widetilde{M}$ and $\xi_1, \xi_2, \xi_3 \in \partial_\infty \widetilde{M}$ such that $\xi_1 \neq \xi_2$, if p is the closest point to x on the geodesic line between ξ_1 and ξ_2, then at least one of the two following cases holds:

- the distance between p and the closest point to x on the geodesic line between ξ_3 and one of ξ_1, ξ_2 is at most c',



- or the distance between the closest point q to ξ_3 on the segment $[x, p]$ and the closest point to x on the geodesic line between ξ_3 and ξ_1 is at most c'.



Note that p, as well as q in the second case, is contained in $\mathscr{C}\Lambda\Gamma$ if $\xi_1, \xi_2, \xi_3 \in \Lambda\Gamma$.

Note that in the second case, if $\widetilde{F}$ and $\widetilde{F} \circ \iota$ are cohomologous via $\widetilde{G}$, with $v_{p,\,q}$ a unit tangent vector at p to the geodesic segment $[p, q]$ (unique if $p \neq q$, with $v_{q,\,p} = -v_{p,\,q}$ if $p = q$), we have, for every $\gamma \in \Gamma$, and by the Γ-invariance of $\widetilde{G}$,

$$\Big| \int_x^{\gamma x} \widetilde{F} - \int_{\gamma x}^{x} \widetilde{F} \Big| = \Big| \int_x^{\gamma x} (\widetilde{F} - \widetilde{F} \circ \iota) \Big| = |\, \widetilde{G}(-v_{\gamma x,\,x}) - \widetilde{G}(v_{x,\,\gamma x})| \leq 2 \max_{v \,\in\, T^1_x \widetilde{M}} |\, \widetilde{G}\,| \;. \quad (31)$$

Note that Formula (28) is also valid, up to enlarging c_5, if p is replaced by any point p' at distance less than c' from p. Indeed, with $F' = F$ or $F' = F \circ \iota$, by Lemma 3.2, the quantity $\big| \int_x^{p'} \widetilde{F}' - \int_x^{p} \widetilde{F}' \big|$ is bounded by a constant depending only on c' and $\kappa = \sup_{\mathscr{N}_{c'}(\mathscr{C}\Lambda\Gamma)} |\, \widetilde{F}\,|$. Furthermore, κ is finite since $|\, \widetilde{F}\,|$ is bounded on $\mathscr{C}\Lambda\Gamma)$ and has at most linear growth by the remark at the end of Subsection 2.1.

Hence, by Equation (28), considering the two above cases (the first one being easier) and using in the second case either Formula (31) or Formula (29), we see that there exists a constant c_6 such that for all $\xi_1, \xi_2, \xi_3 \in \Lambda\Gamma$,

$$D_{F,\,x}(\xi_1, \xi_2) \leq c_6 \max\{D_{F,\,x}(\xi_1, \xi_3), D_{F,\,x}(\xi_2, \xi_3)\} \;.$$

The constant c_6 depends on $\sup_{v \in \pi^{-1}(\mathscr{C}\Lambda\Gamma)} |\widetilde{F}(v)|$, on the Hölder constants of $\widetilde{F}$ and either on $\max_{v \,\in\, T^1_x \widetilde{M}} |\, \widetilde{G}\,|$ if $\widetilde{F}$ and $\widetilde{F} \circ \iota$ are assumed to be cohomologous via $\widetilde{G}$ or on the constant c appearing in Equation (29). Note that, also by Formula (28),

$$D_{F,\,x}(\xi_1, \xi_2) \leq {c_5}^2 D_{F,\,x}(\xi_2, \xi_1) \;.$$

These weak ultrametric triangle inequality and weak symmetry imply (see for instance [GdlH, §7.3]) that there exists $c_7 \in \,]0, 1]$ (depending only on c_5, c_6) such that for every

$\epsilon \in \,]0, c_7]$, there exist a distance $d_{F,\,x,\,\epsilon}$ on $\Lambda\Gamma$ and a constant $c_\epsilon > 0$ (depending only on ϵ, c_5, c_6) such that for all $\xi, \eta \in \Lambda\Gamma$, we have

$$\frac{1}{c_\epsilon}\; D_{F,\,x}(\xi,\eta)^\epsilon \leq d_{F,\,x,\,\epsilon}(\xi,\eta) \leq c_\epsilon\; D_{F,\,x}(\xi,\eta)^\epsilon\;.$$

This proves Lemma 3.7. □

By the above Remark (2), for all $x, y \in \mathscr{C}\Lambda\Gamma$, the distances $d_{F,\,x,\,\epsilon}$ and $d_{F,\,y,\,\epsilon}$ are equivalent. Note that if F is constant, equal to -1, then we may take $c_7 = 1$ and $d_{F,\,x,\,\epsilon} = {d_x}^\epsilon$, where d_x is the visual distance on $\partial_\infty\widetilde{M}$ (see for instance [Bou]).

Note that if in the assumptions of Lemma 3.7, we replace $\mathscr{C}\Lambda\Gamma$ by $\widetilde{M}$, then we get a distance $d_{F,\,x,\,\epsilon}$ on the full boundary $\partial_\infty\widetilde{M}$ instead of just $\Lambda\Gamma$, that satisfies Equation (30).

3.5 The crossratio of (Γ, F)

Let $\partial^4_\infty\widetilde{M}$ be the space of pairwise distinct quadruples of points in $\partial_\infty\widetilde{M}$. The *crossratio* of the potential F is the map from $\partial^4_\infty\widetilde{M}$ to $\mathbb{R}$ defined by

$$[a,b,c,d]_F = \exp\;\frac{1}{2}\lim_{t\to+\infty}\Big(\int_{a_t}^{d_t}\widetilde{F} - \int_{b_t}^{d_t}\widetilde{F} + \int_{b_t}^{c_t}\widetilde{F} - \int_{a_t}^{c_t}\widetilde{F}\Big)\;,$$

where $t\mapsto a_t, t\mapsto b_t, t\mapsto c_t, t\mapsto d_t$ are geodesic rays converging to a, b, c, d respectively (see the figure above Lemma 3.6 on the right hand side). The limit does exist and is independent of the choices of these geodesic rays, again by the Hölder-continuity of F and the properties of asymptotic geodesic rays in $\widetilde{M}$. An easy cancellation argument shows that, for every $x \in \widetilde{M}$,

$$[a,b,c,d]_F = \frac{D_{F,\,x}(a,c)D_{F,\,x}(b,d)}{D_{F,\,x}(a,d)D_{F,\,x}(b,c)}\;. \tag{32}$$

In particular, the right hand side of this equation does not depend on x. We summarise the elementary properties of the crossratio of F in the next statement.

Lemma 3.8 *(i) The crossratio of F is a positive map from $\partial^4_\infty\widetilde{M}$ to $\mathbb{R}$ such that, for every $(a,b,c,d) \in \partial^4_\infty\widetilde{M}$,*

(1) (Γ-invariance) *for every $\gamma\in\Gamma$, we have $[\gamma a,\gamma b,\gamma c,\gamma d]_F = [a,b,c,d]_F$;*

(2) $[a,b,c,d]_F = [b,a,d,c]_F$;

(3) $[a,b,c,d]_F = {[a,b,d,c]_F}^{-1}$;

(4) $[a,b,c,d]_{F\circ\iota} = [c,d,a,b]_F$;

(5) $[a,b,c,d]_F = [a,b,c,\xi]_F\;[a,b,\xi,d]_F$ for every $\xi\in\partial_\infty\widetilde{M}-\{a,b,c,d\}$.

(ii) If the potential F is bounded, then its crossratio is locally Hölder-continuous.

(iii) If $\widetilde{F}' : T^1\widetilde{M}\to\mathbb{R}$ is a Hölder-continuous Γ-invariant map, cohomologous to $\widetilde{F}$, then their associated crossratios coincide:

$$[a,b,c,d]_F = [a,b,c,d]_{F'}\;.$$

(iv) For every hyperbolic element γ in Γ, with attractive and repulsive fixed points γ_- and γ_+ respectively on $\partial_\infty\widetilde{M}$, we have

$$\lim_{n\to+\infty}\frac{1}{n}\log\,[\gamma_-,\gamma^n\xi,\xi,\gamma_+]_F = \frac{1}{2}\big(\mathrm{Per}_F\,\gamma + \mathrm{Per}_F(\gamma^{-1})\big)\;,$$

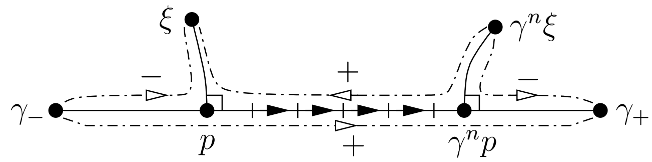
uniformly for ξ in a compact subset of $\partial_\infty\widetilde{M}-\{\gamma_-,\gamma_+\}$.

Assertion (5) is a multiplicative cocycle property in the last two variables of the crossratio of F. Using (4) and (5), it follows that the crossratio of F is also a multiplicative cocycle in the first two variables, that is

$$[a, b, c, d]_F = [a, \xi, c, d]_F \ [\xi, b, c, d]_F$$

for every $\xi \in \partial_\infty \widetilde{M} - \{a, b, c, d\}$.

Proof. The assertions (2), (3) and (4) follow easily from the definition of the crossratio of F. The equations (32) and (26) imply the assertions (1) and (5). Assertion (ii) follows from Equation (32) and Lemma 3.6 (2). The equations (32) and (27) imply Assertion (iii).



To prove the last assertion, define p to be the closest point to ξ on the translation axis Axe_γ of γ. Note that $\gamma^n p$ is then the closest point to $\gamma^n \xi$ on Axe_γ. By hyperbolicity, the four geodesic lines along which one integrates $\widetilde{F}$ to define the crossratio are at bounded distance from the union of Axe_γ and of the geodesic rays $]\xi, p]$ and $]\gamma^n\xi, \gamma^n p]$. Furthermore, the segment between p and $\gamma^n p$ stays very close to the geodesic line between ξ and $\gamma^n \xi$ except for a segment of bounded length near each of its endpoints. By definition of the periods,

$$\int_p^{\gamma^n p} \widetilde{F} = n \operatorname{Per}_F \gamma \quad \text{and} \quad \int_{\gamma^n p}^{p} \widetilde{F} = \int_p^{\gamma^{-n} p} \widetilde{F} = n \operatorname{Per}_F(\gamma^{-1})$$

After dividing by n, the other contributions vanishes as n goes to $+\infty$, by Lemma 3.4 (1) and since $\widetilde{F}$ is bounded on the $\log(1+\sqrt{2})$-neighbourhood of Axe_γ. The result follows. □

Remarks. (1) Let us compare our maps $C_{F,\xi}, D_{F,x}, [\cdot,\cdot,\cdot,\cdot]_F$ with those introduced in Hamenstädt's paper [Ham2] (which assumes that Γ is torsion free, $M = \Gamma\backslash\widetilde{M}$ is compact and F and $F \circ \iota$ are cohomologous, requirements we do not assume in this paper). Let $\zeta : T^1\widetilde{M} \times \mathbb{R} \to \mathbb{R}$ be the map defined by

$$\zeta(v, t) = \int_0^t \widetilde{F}(\phi_s v)\, ds \ .$$

This map is locally Hölder-continuous, invariant under Γ (acting trivially on $\mathbb{R}$) and satisfies $\zeta(v, s+t) = \zeta(v, s) + \zeta(\phi_s(v), t)$. Then Hamenstädt's map $k_\zeta : \widetilde{M} \times \widetilde{M} \times \partial_\infty \widetilde{M} \to \mathbb{R}$ defined in Lemma 1.1 in loc. cit. has the opposite sign to our Gibbs cocycle: for all $x, y \in \widetilde{M}$ and $\xi \in \partial_\infty \widetilde{M}$, we have

$$k_\zeta(x, y, \xi) = -C_{F,\xi}(x, y) \ . \tag{33}$$

Let

$$DT\widetilde{M} = \{(v, w) \in T^1\widetilde{M} \times T^1\widetilde{M} \ : \ \pi(v) = \pi(w) \ \text{ and } \ v \neq w\} \ .$$

Hamenstädt constructs (just above Lemma 1.2 in loc. cit.) a locally Hölder-continuous Γ-invariant map $\alpha_\zeta : DT\widetilde{M} \to \mathbb{R}$, and it may be proved using the equations (25) and (33) that when $(v, w) \in DT\widetilde{M}$ (see the middle picture above Lemma 3.6),

$$\alpha_\zeta(v, w) - \log D_{F,\pi(v)}(v_+, w_+)$$

is uniformly bounded (and even equal to 0 if $F \circ \iota = F$).

(2) Assume in this remark that F and $F \circ \iota$ are cohomologous. It follows from the assertions (4), (iii) and (iv) of the above Lemma 3.8 and from the last equality in Equation (14) that the crossratio of F then satisfies the extra symmetry

$$[a,b,c,d]_F = [c,d,a,b]_F \text{ for every } (a,b,c,d) \in \partial_\infty^4 \widetilde{M}\ ,$$

and that

$$\lim_{n\to+\infty} \frac{1}{n} \log\,[\gamma_-, \gamma^n\xi, \xi, \gamma_+]_F = \mathrm{Per}_F\,\gamma\ .$$

In particular, the crossratio of F then determines the periods of F. By Remark 3.1, the crossratio of the potential function F determines the cohomology class of the restriction of F to the (topological) nonwandering set $\Omega\Gamma$. When Γ is torsion-free and cocompact, this reproves one inclusion in [Ham2, Theo. A].

3.6 The Patterson densities of (Γ, F)

Let $\sigma \in \mathbb{R}$. A *Patterson density* of dimension σ for (Γ, F) is a family of finite nonzero (positive Borel) measures $(\mu_x)_{x\in\widetilde{M}}$ on $\partial_\infty\widetilde{M}$ in the same measure class, such that, for every $\gamma \in \Gamma$, for all $x, y \in \widetilde{M}$, for every $\xi \in \partial_\infty\widetilde{M}$, we have

$$\gamma_*\mu_x = \mu_{\gamma x}\ , \tag{34}$$

$$\frac{d\mu_x}{d\mu_y}(\xi) = e^{-C_{F-\sigma,\,\xi}(x,y)}\ . \tag{35}$$

Note that a Patterson density of dimension σ for (Γ, F) is also a Patterson density of dimension $\sigma + s$ for $(\Gamma, F + s)$, for every $s \in \mathbb{R}$. If $F = 0$, then we get the usual notion of a Patterson density for the group Γ, see for instance [Patt, Sul1, Nic, Bou, Rob1].

Proposition 3.5 implies that the map $x \mapsto \mu_x$ from $\widetilde{M}$ to the space of finite measures on $\partial_\infty\widetilde{M}$, endowed with the weak star topology, is continuous.

The support of the measure μ_x is independent of $x \in \widetilde{M}$ by Equation (35). It is a closed nonempty Γ-invariant subset of $\partial_\infty\widetilde{M}$ by Equation (34). Hence it contains the limit set $\Lambda\Gamma$ of Γ. We will be mostly interested in the case when the support of μ_x is equal to $\Lambda\Gamma$, as this is the case when $\sigma = \delta_{\Gamma,\,F} < +\infty$ and (Γ, F) is of divergence type, by Corollary 5.12. We will prove in Corollary 5.13 that there exists a Patterson density of dimension σ for (Γ, F) whose support contains strictly $\Lambda\Gamma$ if and only if $\Lambda\Gamma \neq \partial_\infty\widetilde{M}$ and the Poincaré series $\sum_{\gamma\in\Gamma} e^{\int_x^{\gamma x}(\widetilde{F}-\sigma)}$ converges. The basic existence result is the following one.

Proposition 3.9 (Patterson) *If $\delta_{\Gamma,\,F} < \infty$, then there exists at least one Patterson density of dimension $\delta_{\Gamma,\,F}$ for (Γ, F), with support exactly equal to $\Lambda\Gamma$.*

Proof. This is proved in [Moh] (see also [Ham2]), following Patterson's construction. Since we will need the construction, we recall it here.

Fix a point y in $\widetilde{M}$. For every $z \in \widetilde{M}$, let $\mathscr{D}_z$ be the unit Dirac mass at z. Let $h : [0, +\infty[\ \to\ ]0, +\infty[$ be a nondecreasing map such that

- for every $\epsilon > 0$, there exists $r_\epsilon \geq 0$ such that $h(t + r) \leq e^{\epsilon t} h(r)$ for all $t \geq 0$ and $r \geq r_\epsilon$;

• if $\overline{Q}_{x,\,y}(s) = \sum_{\gamma\in\Gamma} \; e^{\int_x^{\gamma y}(\widetilde{F}-s)}\; h(d(x,\gamma y))$, then $\overline{Q}_{x,\,y}(s)$ diverges if and only if $s \le \delta_{\Gamma,\,F}$.

If (Γ, F) is of divergence type, we may take $h = 1$ constant. Define the measure

$$\mu_{x,\,s} = \frac{1}{\overline{Q}_{y,\,y}(s)} \sum_{\gamma\in\Gamma} \; e^{\int_x^{\gamma y}(\widetilde{F}-s)}\; h(d(x,\gamma y))\;\mathscr{D}_{\gamma y}$$

on $\widetilde{M}$. It is proved in [Moh] that such a map h exists, and that there exists a sequence $(s_k)_{k\in\mathbb{N}}$ in $]\delta_{\Gamma,\,F}, +\infty[$ converging to $\delta_{\Gamma,\,F}$ such that for every $x \in \widetilde{M}$, the sequence of measures $(\mu_{x,\,s_k})_{k\in\mathbb{N}}$ weak star converges to a measure μ_x on the compact space $\widetilde{M}\cup\partial_\infty\widetilde{M}$, such that $(\mu_x)_{x\in\widetilde{M}}$ is a Patterson density of dimension $\delta_{\Gamma,\,F}$ for (Γ, F). Since $\overline{Q}_{y,\,y}(\delta_{\Gamma,\,F}) = +\infty$ and since the support of $\mu_{x,\,s}$ in $\widetilde{M}\cup\partial_\infty\widetilde{M}$ is equal to $\Gamma y \,\cup\, \Lambda\Gamma$, the support of μ_x is contained in $\Lambda\Gamma$, hence equal to $\Lambda\Gamma$. □

We will often denote by $(\mu_{F,\,x})_{x\in\widetilde{M}}$ a density as in Proposition 3.9. We will prove in Subsection 5.3 that if (Γ, F) is of divergence type, then $(\mu_{F,\,x})_{x\in\widetilde{M}}$ is unique up to a scalar multiple. Note that the definition of this density involves only the normalised potential $F - \delta_{\Gamma,\,F}$. Hence the assumption that $\delta_{\Gamma,\,F}$ is positive is harmless, by Equation (15), and we will often make it in Chapter 9 and Chapter 10.

The main basic tool concerning Patterson densities is the following lemma, proved in [Moh] (see also [Cou2, Lem. 4] with the multiplicative rather than additive convention) along the lines of Sullivan's shadow lemma (see [Rob1, p. 10]), will be very useful. We will prove a more general result in Proposition 11.1 in Subsection 11.1, hence we only refer to [Moh] for a proof of the following result.

Lemma 3.10 (Mohsen's shadow lemma) *Let $\sigma \in \mathbb{R}$, let $(\mu_x)_{x\in\widetilde{M}}$ be a Patterson density of dimension σ for (Γ, F), and let K be a compact subset of $\widetilde{M}$. If R is large enough, there exists $C > 0$ such that for all $\gamma \in \Gamma$ and $x, y \in K$,*

$$\frac{1}{C}\; e^{\int_x^{\gamma y}(\widetilde{F}-\sigma)} \le \mu_x\big(\mathscr{O}_x B(\gamma y, R)\big) \le C\; e^{\int_x^{\gamma y}(\widetilde{F}-\sigma)}\;.$$

The following corollary of Mohsen's shadow lemma shows the inexistence of a Patterson density of dimension less than the critical exponent of (Γ, F). Its proof is similar to the one given by Sullivan [Sul1] in the case $F = 0$ and constant curvature (see also [Rob2, page 147]).

Corollary 3.11 *Let $\sigma \in \mathbb{R}$ and $(\mu_x)_{x\in\widetilde{M}}$ be a Patterson density of dimension σ for (Γ, F).*

(1) For all $x, y \in \widetilde{M}$, there exists $c > 0$ such that for every $n \in \mathbb{N}$, we have

$$\sum_{\gamma\in\Gamma\,:\,n-1<d(x,\gamma y)\le n} e^{\int_x^{\gamma y}(\widetilde{F}-\sigma)} \le c\;.$$

(2) We have $\sigma \ge \delta_{\Gamma,\,F}$.

We will describe in Proposition 5.11 the exact set of elements $\sigma \in \mathbb{R}$ for which there exists a Patterson density of dimension σ for (Γ, F) with support equal to $\Lambda\Gamma$.

Proof. Let $x, y \in \widetilde{M}$. Let R and C be as in Mohsen's shadow lemma 3.10 for $(\mu_x)_{x\in\widetilde{M}}$ and $K = \{x, y\}$. Let $\kappa = \mathrm{Card}\{\gamma \in \Gamma \;:\; d(y, \gamma y) \leq 1+3R\}$, which is finite by discreteness. For every $n \in \mathbb{N}$, let $\Gamma_n = \{\gamma \in \Gamma \;:\; n-1 < d(x, \gamma y) \leq n\}$.

If $\gamma \in \Gamma_n$ and $\xi \in \mathscr{O}_x B(\gamma y, R)$, if p_γ is the closest point to γy on the geodesic ray from x to ξ, then $n - R \leq d(x, p_\gamma) \leq n+1$ by the triangle inequality. Hence, for all $\gamma, \gamma' \in \Gamma_n$, if $\xi \in \mathscr{O}_x B(\gamma y, R) \cap \mathscr{O}_x B(\gamma' y, R)$, then, again by the triangle inequality,

$$d(\gamma' y, \gamma y) \leq d(\gamma' y, p_{\gamma'}) + \big|d(x, p_{\gamma'}) - d(x, p_\gamma)\big| + d(p_\gamma, \gamma y) \leq 1 + 3R\;.$$

Thus, for every $n \in \mathbb{N}$, a point $\xi \in \partial_\infty \widetilde{M}$ belongs to at most κ subsets $\mathscr{O}_x B(\gamma y, R)$ as γ ranges over Γ_n. Therefore

$$\sum_{\gamma\in\Gamma_n} \mu_x\big(\mathscr{O}_x B(\gamma y, R)\big) \leq \kappa\; \mu_x\Big(\bigcup_{\gamma\in\Gamma_n} \mathscr{O}_x B(\gamma y, R)\Big)\;.$$

Now, by the lower bound in Lemma 3.10, and since μ_x is a finite measure,

$$\begin{aligned}\sum_{\gamma\in\Gamma_n} e^{\int_x^{\gamma y}\widetilde{F}} &\leq e^{\sigma(n+1)} \sum_{\gamma\in\Gamma_n} e^{\int_x^{\gamma y}(\widetilde{F}-\sigma)} \leq C\; e^{\sigma(n+1)} \sum_{\gamma\in\Gamma_n} \mu_x\big(\mathscr{O}_x B(\gamma y, R)\big)\\ &\leq C\kappa\, \|\mu_x\|\; e^{\sigma(n+1)}\;. \end{aligned} \tag{36}$$

The first assertion follows. By taking the logarithm, dividing by n and taking the upper limit as n goes to $+\infty$, the second assertion then follows, by the definition of the critical exponent of (Γ, F). $\square$

Here is another consequence of Mohsen's shadow lemma 3.10, concerning the closeness of the gap map defined in Subsection 3.4 to an actual distance.

Proposition 3.12 *Assume that $\delta = \delta_{\Gamma,\,F} < +\infty$. Let $x \in \mathscr{C}\Lambda\Gamma$. Either if $\widetilde{F}$ is reversible and $\widetilde{F}$ is bounded on $\pi^{-1}(\mathscr{C}\Lambda\Gamma)$, or if Γ is convex-cocompact, then for every $\epsilon > 0$ small enough, there exist a distance $d_{F-\delta,\,x,\,\epsilon}$ on $\Lambda\Gamma$ and $c_\epsilon > 0$ such that for all $\xi, \eta \in \Lambda\Gamma$, we have*

$$\frac{1}{c_\epsilon}\; D_{F-\delta,\,x}(\xi,\eta)^\epsilon \leq d_{F-\delta,\,x,\,\epsilon}(\xi,\eta) \leq c_\epsilon\; D_{F-\delta,\,x}(\xi,\eta)^\epsilon\;.$$

Furthermore, if $\sup_{T^1\widetilde{M}} \widetilde{F} < \sigma$, then $d_{F-\delta,\,x,\,\epsilon}$ induces the original topology on $\Lambda\Gamma$.

Proof. The case $\widetilde{F}$ is reversible and $\widetilde{F}$ is bounded on $\pi^{-1}(\mathscr{C}\Lambda\Gamma)$ follows from Lemma 3.7. Assume that Γ is convex-cocompact. Then $\widetilde{F}$ is bounded on $\pi^{-1}(\mathscr{C}\Lambda\Gamma)$, say by $\kappa \geq 0$. Let κ' be the diameter of $\Gamma\backslash\mathscr{C}\Lambda\Gamma$. For all $x, y, z \in \mathscr{C}\Lambda\Gamma$ such that $y \in [x, z]$, let $\alpha, \beta \in \Gamma$ be such that $d(y, \alpha x), d(z, \beta x) \leq \kappa'$. Then by Lemma 3.2 with $r_0 = \kappa'$, there exists $c > 0$ (depending only on κ, κ', the Hölder constants of $\widetilde{F}$ and the bounds on the sectional curvature) such that, with $\widetilde{F}' = \widetilde{F} - \delta$,

$$\begin{aligned}\int_x^z \widetilde{F}' &= \int_x^y \widetilde{F}' + \int_y^z \widetilde{F}'\\ &\leq \int_x^y \widetilde{F}' + \int_{\alpha x}^{\beta x} \widetilde{F}' + \Big|\int_y^z \widetilde{F}' - \int_{\alpha x}^z \widetilde{F}'\Big| + \Big|\int_{\alpha x}^z \widetilde{F}' - \int_{\alpha x}^{\beta x} \widetilde{F}'\Big|\\ &\leq \int_x^y \widetilde{F}' + \int_{\alpha x}^{\beta x} \widetilde{F}' + c \;\leq \int_x^y \widetilde{F}' + \sup_{\gamma\in\Gamma} \int_x^{\gamma x} \widetilde{F}' + c\;.\end{aligned}$$

The existence of a Patterson density of dimension $\sigma = \delta_{\Gamma,\,F} < +\infty$, its finiteness, and the lower bound in Mohsen's shadow lemma imply that for all $x, y \in \widetilde{M}$,

$$\sup_{\gamma\in\Gamma} \int_x^{\gamma y} (\widetilde{F} - \delta) < +\infty \,.$$

This proves that Equation (29) holds for the normalised potential $F - \delta$. Hence, the first claim of Proposition 3.12 follows by Lemma 3.7.

Let us now prove the final claim of Proposition 3.12. The function $\widetilde{F} - \delta$ is bounded from below and from above by a negative constant. Hence by Equation (28), there exist $c, c' > 0$ such that ${d_x}^c \leq D_{F-\delta,\,x} \leq {d_x}^{c'}$ on $\Lambda\Gamma \times \Lambda\Gamma$. on $\pi^{-1}(\mathscr{C}\Lambda\Gamma)$. The claim that the distance $d_{F-\delta,\,x,\,\epsilon}$ on $\Lambda\Gamma$ induces its topology therefore follows from the fact that the visual distance d_x induces the topology on $\partial_\infty \widetilde{M}$. □

Remark. Let $\widetilde{F}^* : T^1\widetilde{M} \to \mathbb{R}$ be a Hölder-continuous Γ-invariant map, which is cohomologous to $\widetilde{F}$ via the map $\widetilde{G} : T^1\widetilde{M} \to \mathbb{R}$. Let $\sigma \in \mathbb{R}$ and let $(\mu_x)_{x\in\widetilde{M}}$ be a Patterson density of dimension σ for (Γ, F). Then the family of measures $(\mu^*_x)_{x\in\widetilde{M}}$ defined by setting, for all $x \in \widetilde{M}$ and $\xi \in \partial_\infty\widetilde{M}$,

$$d\mu^*_x(\xi) = e^{-\widetilde{G}(v_{x\xi})}\, d\mu_x(\xi) \,,$$

where $v_{x\xi}$ is the tangent vector at x of the geodesic ray from x to ξ, is also a Patterson density of dimension σ for (Γ, F^*). Indeed, the invariance property (34) for $(\mu^*_x)_{x\in\widetilde{M}}$ follows from the one for $(\mu_x)_{x\in\widetilde{M}}$ and the invariance of $\widetilde{G}$. And the absolutely continuous property (35) for $(\mu^*_x)_{x\in\widetilde{M}}$ follows from the one for $(\mu_x)_{x\in\widetilde{M}}$ and Equation (23).

3.7 The Gibbs states of (Γ, F)

Let $\sigma \in \mathbb{R}$, let $(\mu^\iota_x)_{x\in\widetilde{M}}$ be a Patterson density of dimension σ for $(\Gamma, F\circ\iota)$, and let $(\mu_x)_{x\in\widetilde{M}}$ be a Patterson density of the same dimension σ for (Γ, F). Using the Hopf parametrisation with respect to any base point x_0 of $\widetilde{M}$, we define the *Gibbs measure on $T^1\widetilde{M}$ associated with the pair of Patterson densities* $\big((\mu^\iota_x)_{x\in\widetilde{M}}, (\mu_x)_{x\in\widetilde{M}}\big)$ (of dimension σ) as the measure $\widetilde{m}$ on $T^1\widetilde{M}$ given by

$$d\,\widetilde{m}(v) = \frac{d\mu^\iota_{x_0}(v_-)\, d\mu_{x_0}(v_+)\, dt}{D_{F-\sigma,\,x_0}(v_-, v_+)^2} \,.$$

Using Equation (25), the following expression of this measure, again in the Hopf parametrisation $v \mapsto (v_-, v_+, t)$ with respect to the base point x_0, will be useful

$$d\,\widetilde{m}(v) = e^{C_{F\circ\iota-\sigma,\,v_-}(x_0,\,\pi(v)) + C_{F-\sigma,\,v_+}(x_0,\,\pi(v))}\; d\mu^\iota_{x_0}(v_-)\, d\mu_{x_0}(v_+)\, dt \,. \tag{37}$$

The measure $\widetilde{m}$ on $T^1\widetilde{M}$ is independent of x_0 by the equations (37), (35) and (21), hence is invariant under the action of Γ by the equations (34) and (26). It is invariant under the geodesic flow. Hence (see Subsection 2.6) it defines a measure m on $T^1M = \Gamma\backslash T^1\widetilde{M}$ which is invariant under the quotient geodesic flow. We call m the *Gibbs measure on T^1M associated with the pair of Patterson densities* $\big((\mu^\iota_x)_{x\in\widetilde{M}}, (\mu_x)_{x\in\widetilde{M}}\big)$. We will justify the terminology in Subsection 3.8.

The (Borel positive) measure

$$d\lambda(\xi,\eta) = \frac{d\mu^{\iota}_{x_0}(\xi)\, d\mu_{x_0}(\eta)}{D_{F-\sigma,\, x_0}(\xi,\eta)^2} \tag{38}$$

on $\partial^2_\infty \widetilde{M}$, which is locally finite and invariant under the diagonal action of Γ on $\partial^2_\infty \widetilde{M}$, is a *geodesic current* for the action of Γ on the Gromov-hyperbolic proper metric space $\widetilde{M}$ in the sense of Ruelle-Sullivan-Bonahon (see for instance [Bon] and references therein).

When $\delta_{\Gamma,\,F} < \infty$, we will denote by $\widetilde{m}_F$, and call the *Gibbs measure on* $T^1\widetilde{M}$ *with potential* F, the Gibbs measure on $T^1\widetilde{M}$ associated with any given pair of densities $\big((\mu_{F\circ\iota,\,x})_{x\in\widetilde{M}}, (\mu_{F,\,x})_{x\in\widetilde{M}}\big)$ (which have the same dimension $\delta_{\Gamma,\,F}$ by Equation (16)). Its induced measure on T^1M will be denoted by m_F and called the *Gibbs measure on* T^1M *with potential* F. By the uniqueness statement in Subsection 5.3, if (Γ, F) is of divergence type (and for instance if m_F is finite), then $\widetilde{m}_F$ and m_F are uniquely defined, up to a scalar multiple.

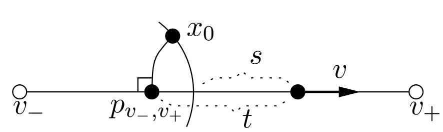
Since the supports of the Patterson densities $(\mu_{F\circ\iota,\,x})_{x\in\widetilde{M}}$ and $(\mu_{F,\,x})_{x\in\widetilde{M}}$ are equal to $\Lambda\Gamma$, the support of $\widetilde{m}_F$ is the set $\widetilde{\Omega}\Gamma$ of tangent vectors to the geodesic lines in $\widetilde{M}$ whose endpoints both lie in $\Lambda\Gamma$. Using the Hopf parametrisation, this support corresponds to $\big((\partial^2_\infty\widetilde{M}) \cap (\Lambda\Gamma\times\Lambda\Gamma)\big)\times\mathbb{R}$. Hence the support of m_F is the (topological) nonwandering set $\Omega\Gamma$ of the geodesic flow in T^1M.

Since only the normalised potential $F - \delta_{\Gamma,\,F}$ is involved in the definition of $\widetilde{m}_F$ and m_F, the assumption that $\delta_{\Gamma,\,F}$ is positive is harmless, by Equation (15). We will often make this assumption in Chapter 9 and Chapter 10.

Remark (1) Since $d\,\widetilde{m}(v) = \frac{d\mu^{\iota}_{x_0}(v_-)\, d\mu_{x_0}(v_+)\, dt}{D_{F\circ\iota-\sigma,\, x_0}(v_+, v_-)^2}$ by Equation (26), the measure $\iota_*\widetilde{m}$ is the Gibbs measure on $T^1\widetilde{M}$ associated with the switched pair of Patterson densities $\big((\mu_x)_{x\in\widetilde{M}}, (\mu^{\iota}_x)_{x\in\widetilde{M}}\big)$ for (F,Γ) and $(F\circ\iota, \Gamma)$ respectively (and similarly on T^1M).

Remark (2) Another parametrisation of $T^1\widetilde{M}$ (used for instance in Subsection 3.9) also depending on the choice of a base point x_0 in $\widetilde{M}$, is the map from $T^1\widetilde{M}$ to $\partial^2_\infty\widetilde{M}\times\mathbb{R}$ sending v to (v_-, v_+, s) where v_- and v_+ are as above and $s = \beta_{v_-}(\pi(v), x_0)$ (we may also use the different time parameter $s = \beta_{v_+}(x_0, \pi(v))$).

For every (η,ξ) in $\partial^2_\infty\widetilde{M}$, let $p_{\eta,\,\xi}$ be the closest point to x_0 on the geodesic line between η and ξ.



For every $v \in T^1\widetilde{M}$, with (v_-, v_+, t) the original the Hopf parametrisation, since

$$s - t = \beta_{v_-}(\pi(v), x_0) - \beta_{v_-}(\pi(v), p_{v_-,v_+}) = \beta_{v_-}(p_{v_-,v_+}, x_0)$$

depends only on v_- and v_+, the measures $d\mu^{\iota}_{x_0}(v_-)\, d\mu_{x_0}(v_+)\, dt$ and $d\mu^{\iota}_{x_0}(v_-)\, d\mu_{x_0}(v_+)\, ds$ are equal. We hence may (and will in the proof of Proposition 3.17) use the second one to define the Gibbs measure associated with the above pair of densities.

Remark (3) It is sometimes useful (see for instance the proof of Theorem 10.4) to consider the following more general construction. Let $x_0, x_0' \in \widetilde{M}$ and $\sigma^\iota, \sigma \in \mathbb{R}$ with possibly $\sigma^\iota \neq \sigma$. Let $(\mu^\iota_x)_{x\in\widetilde{M}}$ be a Patterson density of dimension σ^ι for $(\Gamma, F\circ\iota)$, and let

$(\mu_x)_{x\in\widetilde{M}}$ be a Patterson density of dimension σ for (Γ, F). Using the Hopf parametrisation with respect to x_0 (the original one or the one defined in the previous Remark (2)), we define the *Gibbs measure* associated with the pair of densities $\big((\mu^\iota_x)_{x\in\widetilde{M}}, (\mu_x)_{x\in\widetilde{M}}\big)$ as the measure $\widetilde{m}$ on $T^1\widetilde{M}$ given by

$$d\,\widetilde{m}(v) = e^{C_{F\circ\iota-\sigma^\iota,\, v_-}(x'_0,\, \pi(v)) + C_{F-\sigma,\, v_+}(x_0,\, \pi(v))}\; d\mu^\iota_{x'_0}(v_-)\, d\mu_{x_0}(v_+)\, dt\ . \tag{39}$$

It is again independent of x_0, x'_0 by the equations (35) and (21), hence is invariant under the action of Γ by the equations (34) and (22), and defines a measure m on T^1M, by Subsection 2.6. But it is no longer invariant under the geodesic flow, as it now satisfies, for all $t\in\mathbb{R}$ and $v\in T^1\widetilde{M}$,

$$d(\phi_t)_*\widetilde{m}(v) = e^{C_{F\circ\iota-\sigma^\iota,\, v_-}(\pi(v),\, \pi(\phi_{-t}v)) + C_{F-\sigma,\, v_+}(\pi(v),\, \pi(\phi_{-t}v))}\; d\,\widetilde{m}(v)\ .$$

When F is constant, this simplifies as

$$d(\phi_t)_*\widetilde{m}(v) = e^{(\sigma^\iota-\sigma)t}\; d\,\widetilde{m}(v)\ .$$

When $F=0$, these measures have been considered for instance by Burger [Bur] and Knieper [Kni] in particular cases, and by Roblin [Rob1, page 12] in general, and are sometimes called *Burger-Roblin measures*, see [OhS, Kim]. When F is nonconstant, see for instance [Sch1].

Remark (4) The Gibbs measure m_F is invariant if we replace F by a cohomologous potential. Indeed, let $\widetilde{F}^*: T^1\widetilde{M}\to\mathbb{R}$ be a Hölder-continuous Γ-invariant map cohomologous to $\widetilde{F}$ via the map $\widetilde{G}: T^1\widetilde{M}\to\mathbb{R}$. We have $\delta_{\Gamma,\,F^*} = \delta_{\Gamma,\,F}$ by the remark at the end of Subsection 3.2. Choose the Patterson densities of the remark at the end of Subsection 3.6 to construct $\widetilde{m}_{F^*}$ (such a choice is not important when (Γ, F) is of divergence type, see Subsection 5.3). By Equation (37), by the remark at the end of Subsection 3.6 and by Equation (23), we then have

$$\widetilde{m}_{F^*} = \widetilde{m}_F\ .$$

3.8 The Gibbs property of Gibbs states

Let us indicate why our terminology of Gibbs measures (introduced in Subsection 3.7) is appropriate, by explicitly making an analogy with the symbolic dynamics case and by proving they satisfy the Gibbs property on the dynamical balls of the geodesic flow (see Remark (3) in the beginning of Chapter 10 for other explanations).

Gibbs measures were first introduced in statistical mechanics, and are naturally associated, via the thermodynamic formalism (see for instance [Rue3, Kel, Zin]), to symbolic dynamics. Let us recall the definitions of the pressure of a potential in a general dynamical system, and its equilibrium states, adapting them to the noncompact case, as in [Sar1] for the countable Markov shift case.

If $\phi' = (\phi'_t)_{t\in\mathbb{Z}}$ or $\phi' = (\phi'_t)_{t\in\mathbb{R}}$ is a discrete-time or continuous-time one-parameter group of homeomorphisms of a metric space X' and if $F': X'\to\mathbb{R}$ is a continuous map, define the *pressure* $P(F')$ *of the potential* F' as the upper bound of $h_{m'}(\phi') + \int_{T^1M} F'\, dm'$, on all ϕ'-invariant probability measures m' on X' such that the negative part $\max\{0, -F'\}$ of F' is m'-integrable, where $h_{m'}(\phi') = h_{m'}(\phi'_1)$ is the (metric) entropy of the one-parameter group of homeomorphisms ϕ' with respect to m'. Define an *equilibrium state* of F' as any

such m' realising this upper bound. When $\phi' = \phi$ is the geodesic flow on $X' = T^1M$, we will come back to these definitions in Chapter 6.

Let us now recall what is a Gibbs measure in symbolic dynamics. Let A be a countable set, called an *alphabet* (see [Sar1, Sar3, Sar2] for the infinite case), endowed with the discrete distance $d(a, a') = 1$ if $a \neq a'$, and $d(a, a') = 0$ otherwise. Let Σ be the product topological space $A^{\mathbb{Z}}$, which is compact if A is finite. Let us endow it with the distance (inducing its topology)

$$d((x_i)_{i\in\mathbb{Z}}, (x'_i)_{i\in\mathbb{Z}}) = \sum_{i\in\mathbb{Z}} e^{-i^2} d(x_i, x'_i) \;.$$

Let $\sigma : \Sigma \to \Sigma$ be the (full) *shift map*, which is the homeomorphism of Σ defined by $(x_i)_{i\in\mathbb{Z}} \mapsto (y_i)_{i\in\mathbb{Z}}$ where $y_i = x_{i+1}$ for every $i \in \mathbb{Z}$. Note that if $\pi' : \Sigma \to A$ is the map $(x_i)_{i\in\mathbb{Z}} \mapsto x_0$, then for all $x, x' \in \Sigma$, we have

$$d(x, x') = \sum_{i\in\mathbb{Z}} \; d(\pi'(\sigma^i x), \pi'(\sigma^i x'))\; e^{-i^2}$$

(compare with Equation (5)). For all $n, n' \in \mathbb{N}$ and $a_{-n'}, \dots, a_n$ in A, let

$$[a_{-n'}, \dots, a_n] = \{(x_i)_{i\in\mathbb{Z}} \in \Sigma \;:\; x_i = a_i \;\text{ for } -n' \leq i \leq n\} \;,$$

which is called a *cylinder* in Σ. Recall that given $(x_i)_{i\in\mathbb{Z}} \in \Sigma$, the set of cylinders $\{[x_{-n'}, \dots, x_n] \;:\; n, n' \in \mathbb{N}\}$ is a (natural) neighbourhood basis of $(x_i)_{i\in\mathbb{Z}}$ (even if we take $n' = n$). Finally, let $F : \Sigma \to \mathbb{R}$ be a Hölder-continuous map. For instance when $A = \{-1, +1\}$, the energy map $F : \Sigma \to \mathbb{R}$ defined by $(x_i)_{i\in\mathbb{Z}} \mapsto x_0(x_1 + x_{-1})$ describes the one-dimensional Ising model (see for instance [Kel, page 96]).

The Gibbs measures are then shift-invariant measures, whose mass of a cylinder is a weight defined by integrating the normalised potential along the corresponding piece of orbit. More precisely (compare for instance with [Kel, page 100]), a *Gibbs measure of potential* F is a σ-invariant measure m' on Σ such that there exists $c'(F) \in \mathbb{R}$ such that, for every finite subset A' of A, there exists $c \geq 1$ such that for all $(x_i)_{i\in\mathbb{Z}} \in \Sigma$ and $n, n' \in \mathbb{N}$ with $x_{-n'}, x_n \in A'$, we have

$$\frac{1}{c} \leq \frac{m'([x_{-n'}, \dots, x_n])}{e^{\sum_{i=-n'}^{n} F(\sigma^i x) - (n+n'+1)c'(F)}} \leq c \;. \tag{40}$$

In this symbolic framework, when the alphabet A is finite, with $\phi' = (\sigma^i)_{i\in\mathbb{Z}}$, $X' = \Sigma$ and $F' = F$, it is well-known (see for instance [Bow4, Rue3, Kel, Zin] as well as [HaR]) that a probability Gibbs measure of F exists (with constant $c'(F)$ equal to the pressure of F), is unique, and is the unique equilibrium state of F (see [Sar1, Sar3, Sar2] for what remains true in the countable alphabet case).

The analogy between the geodesic flow $\phi = (\phi_t)_{t\in\mathbb{R}}$ and the full shift $(\sigma^i)_{i\in\mathbb{Z}}$ proceeds as follows. For all $v \in T^1\widetilde{M}$ and $r > 0$, $T, T' \in \mathbb{R}$, define

$$B(v; T, T', r) = \{w \in T^1\widetilde{M} \;:\; \sup_{t\in[-T', T]} \; d(\pi(\phi_t v), \pi(\phi_t w)) < r\} \;,$$

called a *dynamical (or Bowen) ball* around v. They satisfy the following invariance properties: for all $s \in \mathbb{R}$ and $\gamma \in \Gamma$,

$$\phi_s B(v; T, T', r) = B(\phi_s v; T - s, T' + s, r) \quad\text{and}\quad \gamma B(v; T, T', r) = B(\gamma v; T, T', r) \;.$$

Note the following inclusion properties of the dynamical balls: if $r \leq s$, $T \geq S$, $T' \geq S'$, then $B(v; T, T', r)$ is contained in $B(v; S, S', s)$.

Lemma 3.13 *(1) For all $r' \geq r > 0$, there exists $T_{r,r'} \geq 0$ such that for all $T, T' \in \mathbb{R}$, the dynamical ball $B(v; T + T_{r,r'}, T' + T_{r,r'}, r')$ is contained in $B(v; T, T', r)$.*

(2) If $B(v; T, T', r)$ meets $B(u; S, S', r)$ and $T \geq S$, $T' \geq S'$, then v belongs to the dynamical ball $B(u; S, S', 2r)$.

Proof. (1) This follows by the properties of long geodesic segments with endpoints at bounded distance in a CAT(-1) space (see for instance [BrH]).

(2) Let $z \in B(v; T, T', r) \cap B(u; S, S', r)$. By the triangle inequality and since $T \geq S$, $T' \geq S'$, we have

$$\max_{t \in [-S', S]} d(\pi(\phi_t v), \pi(\phi_t u)) \leq \max_{t \in [-T', T]} d(\pi(\phi_t v), \pi(\phi_t z)) + \max_{t \in [-S', S]} d(\pi(\phi_t z), \pi(\phi_t u)) < 2r \;.$$

Hence $v \in B(u; S, S', 2r)$. □

It is easy to see that for every fixed $r > 0$, the set $\{B(v; T, T', r) \;:\; T, T' \geq 0\}$ is a (natural) neighbourhood basis of $v \in T^1\widetilde{M}$ (even if we take $T' = T$), analogous to the set of cylindrical neighbourhoods of a sequence in Σ. We will see in Lemma 3.16 below that these (small) neighbourhoods of v are defined by (small) neighbourhoods of the two points at infinity v_-, v_+ of the geodesic line defined by v, that were mentioned in the introduction. For every $v \in T^1M$, define $B(v; T, T', r')$ as the image by the canonical projection $T^1\widetilde{M} \to T^1M = \Gamma\backslash T^1\widetilde{M}$ of $B(\widetilde{v}; T, T', r')$, for any preimage $\widetilde{v}$ of v in $T^1\widetilde{M}$.

Adapting the definition of [KaH] to the noncompact case, we want to consider the measures on T^1M invariant under the geodesic flow, whose masses of appropriate dynamical balls are weights defined by integrating the normalised potential along the corresponding pieces of orbits, as follows.

Definition 3.14 *A ϕ-invariant measure m' on T^1M* satisfies the Gibbs property *for the potential F with constant $c(F) \in \mathbb{R}$ if for every compact subset K of T^1M, there exist $r > 0$ and $c_{K,r} \geq 1$ such that for all large enough $T, T' \geq 0$, for every $v \in T^1M$ with $\phi_{-T'}v, \phi_T v \in K$, we have*

$$\frac{1}{c_{K,r}} \leq \frac{m'(B(v; T, T', r))}{e^{\int_{-T'}^{T}(F(\phi_t v) - c(F))\,dt}} \leq c_{K,r} \;.$$

The constant $c(F)$ is then uniquely defined. When F is bounded, up to changing the constant $c_{K,r}$, this property does not depend on the constant $r > 0$, by Lemma 3.13.

The following result shows that our terminology of Gibbs measures is indeed appropriate.

Proposition 3.15 *Assume that the potential F is bounded. Let m be the Gibbs measure on T^1M associated with a pair of Patterson densities $\big((\mu^\iota_x)_{x\in\widetilde{M}}, (\mu_x)_{x\in\widetilde{M}}\big)$ of dimension $\delta_{\Gamma,F}$ for $(F \circ \iota, \Gamma)$ and (F, Γ) respectively. Then m satisfies the Gibbs property for the potential F, with constant $c(F) = \delta_{\Gamma,F}$.*

In Chapter 6, we will prove that if the potential F is bounded, then the critical exponent $\delta_{\Gamma,\,F}$ is equal to the pressure $P(\Gamma, F)$, so that $c(F) = P(\Gamma, F)$ in accordance to the case of symbolic dynamics over a finite alphabet.

Proof. We start with the following lemma, which describes the dynamical balls in term of points at infinity.

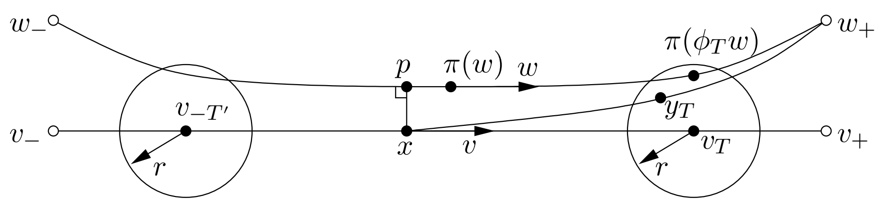
Lemma 3.16 *For all $r, T, T' > 0$, for every $v \in T^1\widetilde{M}$, using the Hopf parametrisation with base point $x = \pi(v)$ and setting $v_T = \pi(\phi_T v)$ and $v_{-T'} = \pi(\phi_{-T'} v)$, we have*

$$B(v; T, T', r) \subset \mathscr{O}_x B(v_{-T'}, 2r) \times \mathscr{O}_x B(v_T, 2r) \times \,]-r, r[\;.$$

Conversely, for every $r > 0$, if $T, T' > 0$ are large enough, then for every $v \in T^1\widetilde{M}$, using the Hopf parametrisation with base point $x = \pi(v)$ and taking $v_{-T'}$ and v_T as above, we have

$$\mathscr{O}_x B(v_{-T'}, r) \times \mathscr{O}_x B(v_T, r) \times \,]-1, 1[\, \subset B(v; T, T', 2r + 2)\;.$$

Proof. To prove the first claim, for every $w \in B(v; T, T', r)$, let p be the closest point to x on the geodesic line defined by w and let y_T be the point at distance T from x on the geodesic ray $[x, w_+[$.



Since the closest point maps do not increase the distances, we have

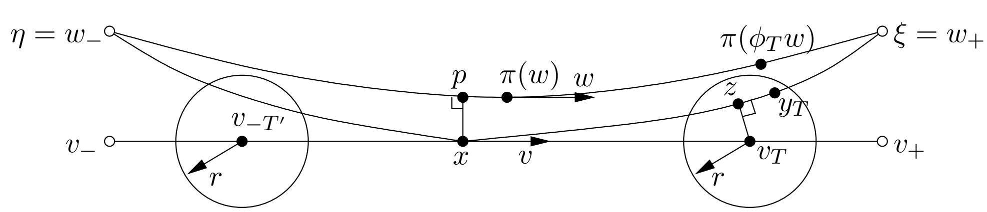
$$d(p, \pi(w)) \le d(\pi(v), \pi(w)) < r\;.$$

By the triangle inequality and by convexity, we have

$$d(y_T, v_T) \le d(y_T, \pi(\phi_T w)) + d(\pi(\phi_T w), \pi(\phi_T v)) \le d(x, \pi(w)) + r < 2r\;,$$

hence w_+ belongs to $\mathscr{O}_x B(v_T, 2r)$. With a similar argument for w_-, this proves the first claim.

To prove the second claim, let $\eta \in \mathscr{O}_x B(v_{-T'}, r)$ and let $\xi \in \mathscr{O}_x B(v_T, r)$, which is different from η if T and T' are larger than a constant depending only on r. Let p be the closest point to x on the geodesic line $]\eta, \xi[$. Since the height of a vertex of a geodesic triangle in a CAT(-1) space whose comparison angle is close to π is small, if T and T' are larger than a constant depending only on r, we have $d(x, p) \le 1$. Let $w \in T^1\widetilde{M}$ be such that $d(p, \pi(w)) < 1$ and $w_- = \eta$, $w_+ = \xi$.



Let us prove that $w \in B(v; T, T', 2r+2)$, which yields the second claim. By convexity, we only have to prove that $d(\pi(\phi_{-T'} w), v_{-T'}) < 2r+2$ and $d(\pi(\phi_T w), v_T) < 2r+2$. Since the argument for the first inequality is similar, we only prove the second inequality.

Let z be the closest point to v_T on the geodesic ray $[x, \xi[$, which satisfies $d(z, v_T) \le r$. Note that $d(x, z) \ge d(x, v_T) - d(v_T, z) \ge T - r$ by the inverse triangle inequality. Let y_T be the point at distance T from x on the geodesic ray $[x, w_+[$, which satisfies

$$d(\pi(\phi_T w), y_T) \le d(\pi(w), x) \le d(\pi(w), p) + d(p, x) < 2$$

by convexity. By the convexity of balls, we have $z \in [x, y_T]$. Then

$$\begin{aligned} d(\pi(\phi_T w), v_T) &\le d(\pi(\phi_T w), y_T) + d(y_T, v_T) < 2 + d(y_T, z) + d(z, v_T) \\ &\le 2 + \big(d(y_T, x) - d(x, z)\big) + r \le 2 + r + T - (T - r) = 2r + 2 \,. \quad \Box \end{aligned}$$

Now, let $\widetilde{m}$ be the Gibbs measure on $T^1\widetilde{M}$ associated with a pair of Patterson densities $\big((\mu^\iota_x)_{x\in\widetilde{M}}, (\mu_x)_{x\in\widetilde{M}}\big)$ of dimension $\delta = \delta_{\Gamma,\,F}$ for $(F\circ\iota, \Gamma)$ and (F, Γ) respectively. Note that $\delta_{\Gamma,\,F} < +\infty$ by Lemma 3.3 (iv), since $\widetilde{F}$ is bounded.

Let $\widetilde{K}$ be a compact subset of $T^1\widetilde{M}$. Let $R_0, C > 0$ be such that the conclusion of Mohsen's shadow lemma 3.10 holds true for $R = R_0$ and $R = 4R_0 + 4$, for the Patterson densities $(\mu^\iota_x)_{x\in\widetilde{M}}$ and $(\mu_x)_{x\in\widetilde{M}}$ and for all x, y in the compact subset $\pi(\widetilde{K})$ of $\widetilde{M}$. Let $r = 2R_0 + 2$. Let us prove that there exists $c' > 0$ such that for all $T, T' > 0$ large enough and for every $v \in T^1\widetilde{M}$ such that $v_T = \pi(\phi_T v) \in \widetilde{K}$ and $v_{-T'} = \pi(\phi_{-T'} v) \in \Gamma\widetilde{K}$, we have

$$\frac{1}{c'} \le \frac{\widetilde{m}(B(v; T, T', r))}{e^{\int_{-T'}^{T} (\widetilde{F}(\phi_t v) - \delta)\, dt}} \le c' \,. \tag{41}$$

Note that $\phi_{-T'} B(v; T, T', r)$ is contained in a compact subset of $T^1\widetilde{M}$ (depending only on $\widetilde{K}$ and r). Hence the multiplicity of the restriction to $\phi_{-T'} B(v; T, T', r)$ of the canonical projection $T^1\widetilde{M} \to T^1 M$ is bounded, by the discreteness of Γ. Since $\widetilde{m}$ is ϕ-invariant, Equation (41) hence implies Proposition 3.15.

By Lemma 3.4 (1) with $r_0 = r$, there exists a constant $c_1 > 0$ such that for all $v \in T^1\widetilde{M}$, $T, T' \ge 0$ and $w \in B(v; T, T', r)$, since $d(x, \pi(w)) \le r$ where $x = \pi(v)$, we have

$$\begin{aligned} &\max\big\{\, |\, C_{F\circ\iota-\delta,\, w_-}(x, \pi(w))\,|,\ |\, C_{F-\delta,\, w_+}(x, \pi(w))\,|\,\big\} \\ &\le\ c_1' =\ c_1\, e^r\ +\ r \max_{T^1\widetilde{M}} |\widetilde{F} - \delta| \,. \end{aligned}$$

Note that c_1' is finite since $\widetilde{F}$ is bounded. We use the Hopf parametrisation with respect to the base point x. By Equation (37) and the first claim in Lemma 3.16, we hence have

$$\widetilde{m}(B(v; T, T', r)) \le 2\, r\, e^{2c_1'}\ \mu^\iota_x(\mathscr{O}_x B(v_{-T'}, 2r))\ \mu_x(\mathscr{O}_x B(v_T, 2r)).$$

Since $2r = 4R_0 + 4$ and $v_T, v_{-T'} \in \Gamma\widetilde{K}$, Mohsen's shadow lemma 3.10 and Equation (13) imply that

$$\begin{aligned} \widetilde{m}(B(v; T, T', r)) &\le 2\, r\, e^{2c_1'}\, C^2\ e^{\int_x^{v_{-T'}} (\widetilde{F}\circ\iota - \delta)}\ e^{\int_x^{v_T} (\widetilde{F} - \delta)} \\ &= 2\, r\, e^{2c_1'}\, C^2\ e^{\int_{-T'}^{T} (\widetilde{F}(\phi_t v) - \delta)\, dt} \,. \end{aligned}$$

The proof of the lower bound in Equation (41) is similar, using the second claim in Lemma 3.16. This proves Proposition 3.15. □

Remark. A more usual definition of the dynamical balls is the following one. For every $T \geq 0$, consider the Γ-invariant distance $d'_{\phi,T}$ on $T^1\widetilde{M}$ defined by

$$\forall\, v, w \in T^1\widetilde{M}, \quad d'_{\phi,T}(v,w) = \max_{t\in[0,T]} d'(\phi_t v, \phi_t w)\;,$$

where $d' = d'_{T^1\widetilde{M}}$ is the distance on $T^1\widetilde{M}$ defined in Equation (7). For all $\epsilon > 0$ and $T \geq 0$, let

$$B'(v;\epsilon,T) = \{w \in T^1\widetilde{M}\;:\; \max_{t\in[0,T]} d'(\phi_t v, \phi_t w)\; < \epsilon\}$$

be the open ball with center $v \in T^1\widetilde{M}$ and radius ϵ for this distance $d'_{\phi,T}$. By the definition of d', we have

$$B'(v;\epsilon,T) = B'(v;T,-1,\epsilon)\;.$$

Note that if $d' = d'_{T^1\widetilde{M}}$ is replaced by any Hölder-equivalent distance d'' (for instance the Riemannian distance d_S induced by the Sasaki metric, or the distance $d = d_{T^1\widetilde{M}}$ defined in Equation (5)), and if $B''(v;\epsilon,T)$ are the corresponding dynamical balls, then there exist $\epsilon_0, c > 0$ and $\alpha \in\,]0,1]$ such that, for all $\epsilon \in\,]0,\epsilon_0]$ and $T \geq 0$,

$$B'(v;\frac{1}{c}\,\epsilon^{\frac{1}{\alpha}},T) \subset B''(v;\epsilon,T) \subset B'(v;c\,\epsilon^{\alpha},T)\;.$$

Since $\widetilde{M}$ has pinched negative curvature and by the divergence properties of the geodesic lines, for all $r > 0$, there exists $\epsilon_r, a_r, b_r, c_c, T_r > 0$ such that for all $\epsilon \in\,]0,\epsilon_r]$ and $T \geq T_r$,

$$B(v;T-a_r\log\epsilon+c_c,-a_r\log\epsilon+c_c,r) \subset B'(v;\epsilon,T) \subset B(v;T-b_r\log\epsilon-c_c,-b_r\log\epsilon-c_c,r)\;.$$

When $\widetilde{M}$ has constant curvature -1, we may take $a_r = b_r = 1$. But in general variable curvature, using the dynamical balls $B'(v;\epsilon,T)$ instead of the ones we are using would give a less precise control.

3.9 Conditional measures of Gibbs states on strong (un)stable leaves

The aim of this subsection is to describe the conditional measures of the Gibbs measures constructed in Subsection 3.7 on the leaves of the strong unstable and strong stable foliations. We refer for instance to [DeM, Chap. 3, §70] for general information on the disintegration of measures.

We start by describing the family of measures on the strong unstable and strong stable leaves which will allow us to determine the conditional measures of the Gibbs measures.

Let $\sigma \in \mathbb{R}$. A Patterson density $(\mu_x)_{x\in\widetilde{M}}$ of dimension σ for (Γ, F) defines a family of measures $(\mu_{W^{su}(v)})_{v\in T^1\widetilde{M}}$ on the strong unstable leaves, in the following way. Recall that for every $v \in T^1\widetilde{M}$, the map $w \mapsto w_+$ from $W^{su}(v)$ to $\partial_\infty\widetilde{M} - \{v_-\}$ is a homeomorphism. This allows us to define a measure $\mu_{W^{su}(v)}$ on $W^{su}(v)$ by

$$d\mu_{W^{su}(v)}(w) = e^{C_{F-\sigma,\,w_+}(x_0,\,\pi(w))}\, d\mu_{x_0}(w_+)\;, \tag{42}$$

where x_0 is any point of $\widetilde{M}$. This measure is nonzero, since Γ being nonelementary, the support of μ_{x_0} is not reduced to $\{v_-\}$. By the absolute continuity property (35) of $(\mu_x)_{x\in\mathbb{R}}$ and by the cocycle property (21) of the Gibbs cocycle $C_{F-\sigma}$, the measure $\mu_{W^{su}(v)}$ does not depend on the choice of x_0. By the invariance properties (34) and (22), the family of measures $(\mu_{W^{su}(v)})_{v\in T^1\widetilde{M}}$ is Γ-equivariant: for all $v\in T^1\widetilde{M}$ and $\gamma\in\Gamma$,

$$\gamma_*\mu_{W^{su}(v)} = \mu_{W^{su}(\gamma v)}\ . \tag{43}$$

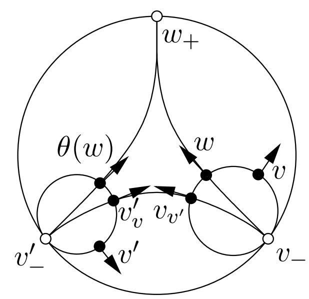
Note that $\mu_{W^{su}(v)} = \mu_{W^{su}(v')}$ if v and v' are in the same strong unstable leaf. If the support of μ_x is $\Lambda\Gamma$, the support of $\mu_{W^{su}(v)}$ is

$$\operatorname{Supp}(\mu_{W^{su}(v)}) = \{w\in W^{su}(v)\ :\ w_+\in\Lambda\Gamma\}\ . \tag{44}$$

We will also consider the measure $\mu_{W^{su}(v)}$ on $W^{su}(v)$ as a measure on $T^1\widetilde{M}$ with support contained in $W^{su}(v)$. The map $v\mapsto\mu_{W^{su}(v)}$ is continuous for the weak star topology on the space of measures on $T^1\widetilde{M}$. The family of measures $(\mu_{W^{su}(v)})_{v\in T^1\widetilde{M}}$ uniquely determines the Patterson density $(\mu_x)_{x\in\widetilde{M}}$, by Equation (42). The geodesic flow $(\phi_t)_{t\in\mathbb{R}}$ preserves the measure class of these measures: for all $t\in\mathbb{R}$ and $w\in W^{su}(v)$, we have

$$\frac{d\,(\phi_t)_*\mu_{W^{su}(v)}}{d\,\mu_{W^{su}(\phi_t v)}}(\phi_t w) = e^{C_{F-\sigma,\,w_+}(\pi(\phi_t w),\,\pi(w))} = e^{\int_0^t(\widetilde{F}(\phi_s w)-\sigma)\,ds}\ , \tag{45}$$

(which is equal to $e^{\int_{\pi(w)}^{\pi(\phi_t w)}(\widetilde{F}-\sigma)}$ if $t\geq 0$ and to $e^{-\int_{\pi(\phi_t w)}^{\pi(w)}(\widetilde{F}-\sigma)}$ otherwise, by Equation (18)).



For all $v,v'\in T^1\widetilde{M}$ such that $v_-\neq v'_-$, let $v_{v'}$ be the unique element of $W^{su}(v)$ such that $(v_{v'})_+ = v'_-$. The map $\theta^{su}=\theta^{su}_{v',v}: W^{su}(v)-\{v_{v'}\}\to W^{su}(v')-\{v'_v\}$ sending w to the unique element $\theta^{su}(w)$ such that $\theta^{su}(w)_+ = w_+$, is a homeomorphism. An easy computation using Equation (42) and Equation (21) shows that for every $w\in W^{su}(v)-\{v_{v'}\}$, we have

$$\frac{d\,(\theta^{su})_*\mu_{W^{su}(v)}}{d\,\mu_{W^{su}(v')}}(\theta^{su}(w)) = e^{C_{F-\sigma,\,w_+}(\pi(\theta^{su}(w)),\,\pi(w))}\ . \tag{46}$$

Conversely, a family of nonzero measures $(\mu_{W^{su}(v)})_{v\in T^1\widetilde{M}}$ on $T^1\widetilde{M}$, constant on the strong unstable leaves, with $\mu_{W^{su}(v)}$ supported in $W^{su}(v)$, satisfying the properties (43), (45) and (46) defines a unique Patterson density $(\mu_x)_{x\in\widetilde{M}}$ of dimension σ for (Γ,F) satisfying Equation (42).

By the equivariance property (43), the family $(\mu_{W^{su}(v)})_{v\in T^1\widetilde{M}}$ induces on T^1M a weak star continuous family of measures, that we will denote by $(\mu_{W^{su}(v)})_{v\in T^1M}$, constant on the strong unstable leaves, with $\mu_{W^{su}(v)}$ supported in $W^{su}(v)$ for every $v\in T^1M$.

A Patterson density $(\mu_x)_{x\in\widetilde{M}}$ of dimension σ for (Γ,F) also defines a family of measures $(\mu_{W^u(v)})_{v\in T^1\widetilde{M}}$ on the unstable leaves, in the following way. Let $x_0\in\widetilde{M}$ and $v\in T^1\widetilde{M}$. We have a homeomorphism from $W^{\rm su}(v)\times\mathbb{R}$ to $W^{\rm u}(v)$, defined by $(w,t)\mapsto w'=\phi_t w$, whose inverse is the map

$$w'\mapsto\big(w=\phi_{\beta_{v_-}(\pi(v),\,\pi(w'))}w',\,t=\beta_{v_-}(\pi(w'),\pi(v))\big)\ .$$

For every $\gamma \in \Gamma$, the following diagram commutes :

$$\begin{array}{ccc} W^{\rm su}(v) \times \mathbb{R} & \to & W^{\rm u}(v) \\ {\scriptstyle \gamma\times{\rm id}}\downarrow & & \downarrow{\scriptstyle \gamma} \\ W^{\rm su}(\gamma v) \times \mathbb{R} & \to & W^{\rm u}(\gamma v)\ . \end{array}$$

Using this homeomorphism, as well as the homeomorphism from the unstable leaf $W^{\rm u}(v)$ to $(\partial_\infty \widetilde{M} - \{v_-\}) \times \mathbb{R}$ defined by

$$w' \mapsto \big(w'_+, t = \beta_{v_-}(\pi(w'), \pi(v))\big)\ ,$$

we may define a nonzero (positive Borel) measure $\mu_{W^u(v)}$ on $W^{\rm u}(v)$, by

$$\begin{aligned} d\mu_{W^u(v)}(w') &= e^{C_{F-\sigma,\, w_+}(\pi(w),\, \pi(\phi_t w))}\ d\mu_{W^{su}(v)}(w)dt \\ &= e^{C_{F-\sigma,\, w'_+}(x_0,\, \pi(w'))}\ d\mu_{x_0}(w'_+)dt\ . \end{aligned} \tag{47}$$

In particular, the second formula is independent of the choice of x_0. Note that when $\widetilde{F} = 0$, the first equality simplifies as

$$d\mu_{W^u(v)}(w') = e^{-\sigma\, t}\ d\mu_{W^{su}(v)}(w)dt\ .$$

By the invariance property (22) of the Gibbs cocycle and by Equation (43), the family of measures $(\mu_{W^u(v)})_{v\in T^1\widetilde{M}}$ is equivariant under Γ: for all $v \in T^1\widetilde{M}$ and $\gamma \in \Gamma$,

$$\gamma_* \mu_{W^u(v)} = \mu_{W^u(\gamma v)}\ . \tag{48}$$

Note that $\mu_{W^u(v)} = \mu_{W^u(v')}$ if v and v' are in the same unstable leaf, by Equation (47) and since the parametrisation of $W^s(v)$ by $(\partial_\infty \widetilde{M} - \{v_-\}) \times \mathbb{R}$ is only changed, when passing from v to v', by adding a constant to the parameter t, as the Lebesgue measure dt is invariant under translations. If the support of μ_x is $\Lambda\Gamma$, the support of $\mu_{W^u(v)}$ is

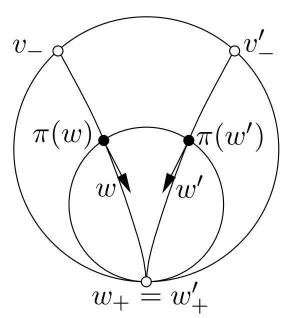
$${\rm Supp}(\mu_{W^u(v)}) = \{w \in W^u(v)\ :\ w_+ \in \Lambda\Gamma\}\ . \tag{49}$$

The map $v \mapsto \mu_{W^u(v)}$ is continuous for the weak star topology on the space of measures on $T^1\widetilde{M}$. The family of measures $(\mu_{W^u(v)})_{v\in T^1\widetilde{M}}$ uniquely determines the family of measures $(\mu_{W^{su}(v)})_{v\in T^1\widetilde{M}}$, hence the Patterson density $(\mu_x)_{x\in\widetilde{M}}$. The geodesic flow $(\phi_t)_{t\in\mathbb{R}}$ preserves the measure class of these measures: for all $t \in \mathbb{R}$ and $w \in W^u(v)$, we have

$$\frac{d\,(\phi_t)_*\mu_{W^u(v)}}{d\,\mu_{W^u(v)}}(\phi_t w) = e^{\int_0^t (\widetilde{F}(\phi_s w) - \sigma)\, ds}\ . \tag{50}$$

For all $v, v' \in T^1\widetilde{M}$, the map $\theta^u = \theta^u_{v',v} : W^u(v) - \{w \in W^u(v)\ :\ w_+ = v'_-\} \to W^u(v') - \{w' \in W^u(v')\ :\ w'_+ = v_-\}$ sending w to the unique element $w' = \theta^u(w)$ in $W^{ss}(w)$ such that $w'_- = v'_-$ is a homeomorphism. An easy computation using Equation (46) shows that for every $w \in W^u(v)$ with $w_+ \neq v'_-$, we have

$$\frac{d\,(\theta^u)_*\mu_{W^u(v)}}{d\,\mu_{W^u(v')}}(\theta^u(w)) = e^{C_{F-\sigma,\, w_+}(\pi(\theta^u(w)),\, \pi(w))}\ . \tag{51}$$



Similarly, given $\sigma^\iota\in\mathbb{R}$ and a Patterson density $(\mu^\iota_x)_{x\in\widetilde{M}}$ of dimension σ^ι for $(\Gamma, F\circ\iota)$, we define weak star continuous families $(\mu^\iota_{W^{ss}(v)})_{v\in T^1\widetilde{M}}$ and $(\mu^\iota_{W^{s}(v)})_{v\in T^1\widetilde{M}}$ of nonzero measures on $T^1\widetilde{M}$, constant on respectively the strong stable leaves and stable leaves, equivariant under Γ, as follows:

- using the homeomorphism from $W^{ss}(v)$ to $\partial_\infty\widetilde{M}-\{v_+\}$ defined by $w\mapsto w_-$, we set

$$d\mu^\iota_{W^{ss}(v)}(w)=e^{C_{F\circ\iota-\sigma^\iota,\,w_-}(x_0,\,\pi(w))}\,d\mu^\iota_{x_0}(w_-)\;, \tag{52}$$

which is independent of $x_0\in\widetilde{M}$,

- using the homeomorphism from $W^{\rm ss}(v)\times\mathbb{R}$ to $W^{\rm s}(v)$, defined by $(w,t)\mapsto w'=\phi_t w$, and the homeomorphism from $W^{\rm s}(v)$ to $(\partial_\infty\widetilde{M}-\{v_+\})\times\mathbb{R}$ defined by $w'\mapsto \big(w'_-, t=\beta_{v_+}(\pi(v),\pi(w'))\big)$, we set

$$\begin{aligned} d\mu^\iota_{W^{s}(v)}(w')&=e^{C_{F\circ\iota-\sigma^\iota,\,w_-}(\pi(w),\,\pi(\phi_t w))}\,d\mu^\iota_{W^{ss}(v)}(w)dt\\ &=e^{C_{F\circ\iota-\sigma^\iota,\,w'_-}(x_0,\,\pi(w'))}\,d\mu^\iota_{x_0}(w'_-)dt\;. \end{aligned}\tag{53}$$

As above, for all $v\in T^1\widetilde{M}$ and $\gamma\in\Gamma$, we have

$$\gamma_*\mu^\iota_{W^s(v)}=\mu^\iota_{\gamma W^s(v)}\;. \tag{54}$$

As above, if the support of μ^ι_x is $\Lambda\Gamma$, then the support of $\mu^\iota_{W^{ss}(v)}$ is $\{w\in W^{ss}(v)\;:\;w_-\in\Lambda\Gamma\}$, and, for every $t\in\mathbb{R}$,

$$\forall\, w\in W^{ss}(v),\quad \frac{d\,(\phi_t)_*\mu^\iota_{W^{ss}(v)}}{d\,\mu^\iota_{W^{ss}(\phi_t v)}}(\phi_t w)=e^{\int_0^t(\widetilde{F}\circ\iota(\phi_s w)-\sigma^\iota)\,ds}\;,$$

$$\forall\, w\in W^{s}(v),\quad \frac{d\,(\phi_t)_*\mu^\iota_{W^{s}(v)}}{d\,\mu^\iota_{W^{s}(v)}}(\phi_t w)=e^{\int_0^t(\widetilde{F}\circ\iota(\phi_s w)-\sigma^\iota)\,ds}\;.$$

For all $v,v'\in T^1\widetilde{M}$, the map $\theta^s:\{w\in W^s(v)\;:\;w_-\neq v'_+\}\to\{w'\in W^s(v')\;:\;w'_-\neq v_+\}$ sending w to the unique element w' in $W^{su}(w)\cap W^s(v')$, is a homeomorphism. For every w in the domain of θ^s, we have

$$\frac{d\,(\theta^s)_*\mu^\iota_{W^s(v)}}{d\,\mu^\iota_{W^s(v')}}(\theta^s(w))=e^{C_{F\circ\iota-\sigma^\iota,\,w_-}(\pi(\theta^s(w)),\,\pi(w))}\;. \tag{55}$$

Now, before determining the conditional measures of the Gibbs measures, we describe the local product structures of $T^1\widetilde{M}$ defined by the strong unstable and strong stable foliations, giving the local fibrations allowing to disintegrate the Gibbs measures.

Fix $w\in T^1\widetilde{M}$, and let

$$U_w=T^1\widetilde{M}-(W^s(-w)\cup W^u(-w))=\{v\in T^1\widetilde{M}\;:v_+\neq w_-, v_-\neq w_+\}$$

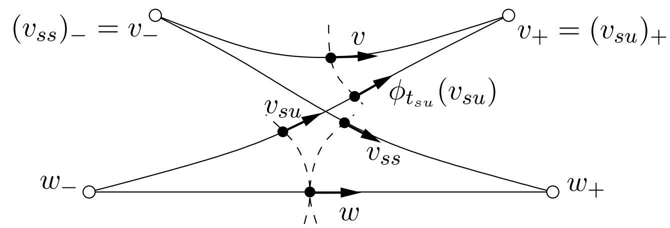
be the open neighbourhood of w in $T^1\widetilde{M}$ consisting of the unit tangent vectors that belong neither to the stable nor to the unstable leaf of $-w$. Note that U_w is dense $T^1\widetilde{M}$. For every $v\in U_w$,

• let v_{su} be the intersection point of $W^{su}(w)$ and $W^s(v)$, which is the unique point v_{su} of $W^{su}(w)$ such that $(v_{su})_+ = v_+$,

• let v_{ss} be the intersection point of $W^{ss}(w)$ and $W^u(v)$, which is the unique point v_{ss} of $W^{ss}(w)$ such that $(v_{ss})_- = v_-$,

• and let $t_{su} \in \mathbb{R}$ be the unique real number such that $\phi_{t_{su}}(v_{su}) \in W^{ss}(v)$.

Then the map $v \mapsto (v_{ss}, v_{su}, t_{su})$ from U_w to $W^{ss}(w) \times W^{su}(w) \times \mathbb{R}$ is a homeomorphism. Note that, when v_- and v_+ are fixed, so is $(v_{su})_+$, and the difference of the time parameters $t - t_{su}$ between the Hopf parametrisation and this one is constant.



The image by this homeomorphism of the Gibbs measures are absolutely continuous with respect to product measures. Indeed, let $\sigma \in \mathbb{R}$ and let $\widetilde{m}$ be the Gibbs measure on $T^1\widetilde{M}$ associated with a pair of Patterson densities $(\mu^\iota_x)_{x\in\widetilde{M}}$ and $(\mu_x)_{x\in\widetilde{M}}$ of dimension σ for $(\Gamma, F \circ \iota)$ and (Γ, F) respectively. Fix $x \in \widetilde{M}$. By definition of the measures on the strong unstable and stable leaf, we have

$$d\mu_{W^{su}(w)}(v_{su}) = e^{C_{F-\sigma,\,(v_{su})_+}(x,\,\pi(v_{su}))}\; d\mu_x((v_{su})_+)$$

and

$$d\mu^\iota_{W^{ss}(w)}(v_{ss}) = e^{C_{F\circ\iota-\sigma,\,(v_{ss})_-}(x,\,\pi(v_{ss}))}\; d\mu^\iota_x((v_{ss})_-)\;.$$

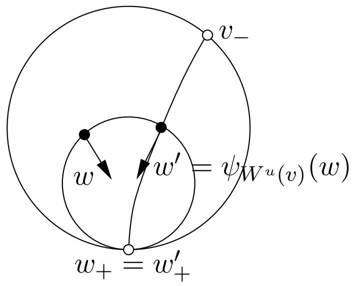
By Equation (37) and the cocycle property, for all $v \in U_w$, we hence have

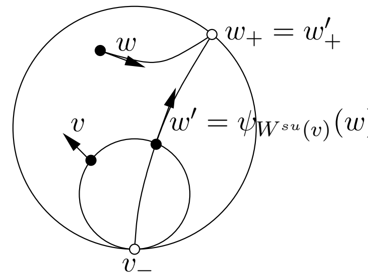
$$d\widetilde{m}(v) = e^{C_{F\circ i-\sigma,\,v_-}(\pi(v_{ss}),\,\pi(v))+C_{F-\sigma,\,v_+}(\pi(v_{su}),\,\pi(v))}\; d\mu^\iota_{W^{ss}(w)}(v_{ss})\,d\mu_{W^{su}(w)}(v_{su})\,dt_{su}\;. \tag{56}$$

This quasi-product property, either on the form of Equation (37) or of Equation (56), is crucial in our paper. In particular, if a measurable subset A of $W^{su}(w)$ has measure 0 for $\mu_{W^{su}(w)}$, then the measurable set $\bigcup_{v\in A} W^s(v)$ has measure 0 for $\widetilde{m}$.

A slightly different way of understanding this property is as follows. Let us fix $v \in T^1\widetilde{M}$. Let us define $U_{v_-} = \{w \in T^1\widetilde{M}\ :\ w_+ \neq v_-\}$, which is an open and dense subset of $T^1\widetilde{M}$, invariant under the geodesic flow. If $\mu_x(\{v_-\}) = 0$ (this does not depend on $x \in \widetilde{M}$), for instance if the measure μ_x has no atom, then U_{v_-} has full measure with respect to $\widetilde{m}$. We have $U_{\gamma v_-} = \gamma U_{v_-}$ for every isometry γ of $\widetilde{M}$, and in particular U_{v_-} is invariant under the isometries of $\widetilde{M}$ fixing v_-.

The map $\psi_{W^u(v)}$ from U_{v_-} to $W^u(v)$, sending $w \in U_{v_-}$ to the unique element w' in $W^{ss}(w) \cap W^u(v)$, is a continuous fibration over the unstable leaf $W^u(v)$, whose fiber over $w' \in W^u(v)$ is precisely the strong stable leaf $W^{ss}(w')$ of w' (see the left hand picture below). This map depends only on the unstable leaf of v. For every isometry γ of $\widetilde{M}$, we have $\psi_{W^u(\gamma v)} \circ \gamma = \gamma \circ \psi_{W^u(v)}$. For every $t \in \mathbb{R}$, we have $\psi_{W^u(v)} \circ \phi_t = \phi_t \circ \psi_{W^u(v)}$: the fibration $\psi_{W^u(v)}$ commutes with the geodesic flow.





Similarly, for every $v \in T^1\widetilde{M}$, the map $\psi_{W^{su}(v)}$ from U_{v_-} to $W^{su}(v)$, sending $w \in U_{v_-}$ to the unique element w' in $W^s(w) \cap W^{su}(v)$, is a continuous fibration over the strong unstable leaf $W^{su}(v)$, whose fiber over $w' \in W^{su}(v)$ is precisely the stable leaf $W^s(w')$ of w' (see the left hand picture below). For every isometry γ of $\widetilde{M}$, we have $\psi_{W^{su}(\gamma v)} \circ \gamma = \gamma \circ \psi_{W^{su}(v)}$. For every $t \in \mathbb{R}$, we have $\psi_{W^{su}(v)} \circ \phi_t = \psi_{W^{su}(v)}$: the fibration $\psi_{W^{su}(v)}$ is invariant under the geodesic flow.

The aforementioned desintegration result of the Gibbs measures on the strong unstable foliation is the following one.

Proposition 3.17 *Let $\widetilde{m}$ be the Gibbs measure on $T^1\widetilde{M}$ associated with a pair of Patterson densities $(\mu^\iota_x)_{x\in\widetilde{M}}$ and $(\mu_x)_{x\in\widetilde{M}}$ of dimensions σ^ι and σ for $(\Gamma, F\circ\iota)$ and (Γ, F), respectively. Let $v \in T^1\widetilde{M}$.*

(1) The restriction to U_{v_-} of the measure $\widetilde{m}$ disintegrates by the fibration $\psi_{W^u(v)}$ over the measure $\mu_{W^u(v)}$, with conditional measure on the fiber $W^{ss}(w')$ of w' the measure $e^{C_{F,\,w'_+}(\pi(w'),\,\pi(w))}\;d\mu^\iota_{W^{ss}(w')}(w)$: for every $w \in T^1\widetilde{M}$ such that $w_+ \neq v_-$, we have

$$d\widetilde{m}(w) = \int_{w'\in W^u(v)} e^{C_{F,\,w'_+}(\pi(w'),\,\pi(w))}\;d\mu^\iota_{W^{ss}(w')}(w)\;d\mu_{W^u(v)}(w')\;.$$

(2) The restriction to U_{v_-} of the measure $\widetilde{m}$ disintegrates by the fibration $\psi_{W^{su}(v)}$ over the measure $\mu_{W^{su}(v)}$, with conditional measure on the fiber $W^s(w')$ of w' the measure $e^{C_{F,\,w'_+}(\pi(w'),\,\pi(w))}\;d\mu^\iota_{W^s(w')}(w)$: for every $w \in T^1\widetilde{M}$ such that $w_+ \neq v_-$, we have

$$d\widetilde{m}(w) = \int_{w'\in W^{su}(v)} e^{C_{F,\,w'_+}(\pi(w'),\,\pi(w))}\;d\mu^\iota_{W^s(w')}(w)\;d\mu_{W^{su}(v)}(w')\;.$$

We leave to the reader the analogous statements obtained by exchanging the stable and unstable foliations. When $F = 0$ and $(\mu^\iota_x)_{x\in\widetilde{M}} = (\mu_x)_{x\in\widetilde{M}}$, we recover the well-known fact, due to Margulis when M is compact, that $\mu_{W^{su}(v)}$ and $\mu_{W^{ss}(v)}$ are the conditional measures of $\widetilde{m}$ along the leaves of the foliations $\widetilde{\mathscr{W}}^{su}$ and $\widetilde{\mathscr{W}}^{ss}$, respectively.

Proof. Fix $v \in T^1\widetilde{M}$. For every continuous map $\varphi \in \mathscr{C}_c(U_{v_-};\mathbb{R})$ with compact support in the domain U_{v_-} of $\psi_{W^u(v)}$, let

$$I_\varphi = \int_{w\in T^1\widetilde{M}} \varphi(w)\;d\widetilde{m}(w) = \int_{w\in U_{v_-}} \varphi(w)\;d\widetilde{m}(w)\;.$$

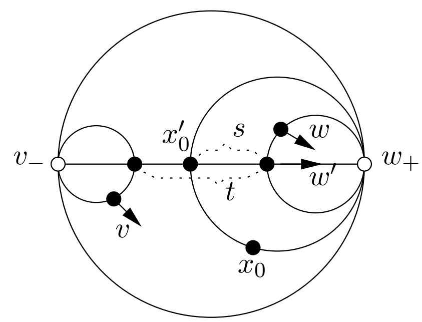
Using the Hopf parametrisation $w \mapsto \big(w_-, w_+, s = \beta_{w_+}(x_0, \pi(w))\big)$ as in Remark (2) of Subsection 3.7, and Equation (39) with $x_0' = x_0$ fixed in $\widetilde{M}$, we have

$$I_\varphi = \int_{w_+ \in \partial_\infty \widetilde{M} - \{v_-\}} \int_{w_- \in \partial_\infty \widetilde{M} - \{w_+\}} \int_{s \in \mathbb{R}} \varphi(w)\, e^{C_{F \circ \iota - \sigma^\iota,\, w_-}(x_0, \pi(w)) + C_{F-\sigma,\, w_+}(x_0, \pi(w))}$$
$$d\mu^\iota_{x_0}(w_-)\, d\mu_{x_0}(w_+)\, ds\ .$$

For every $w \in U_{v_-}$, let $w' = \psi_{W^u(v)}(w)$ be the unique point in $W^{ss}(w) \cap W^u(v)$. By the cocycle property of the Gibbs cocycle, and since $\pi(w)$ and $\pi(w')$ are in the same horosphere centered at $w_+ = w'_+$, we have

$$\begin{aligned} C_{F-\sigma,\, w_+}(x_0, \pi(w)) - C_{F-\sigma,\, w'_+}(x_0, \pi(w')) &= C_{F-\sigma,\, w'_+}(\pi(w'), \pi(w)) \\ &= C_{F,\, w'_+}(\pi(w'), \pi(w))\ . \end{aligned}$$

When $w_+ = w'_+$ is fixed, the time parameter $s = \beta_{w_+}(x_0, \pi(w)) = \beta_{w_+}(x_0, \pi(w'))$ differs from the time parameter $t = \beta_{v_-}(\pi(w'), \pi(v))$ by a constant (equal to $\beta_{v_-}(\pi(v), x'_0)$ where x'_0 is the intersection point of the geodesic line $]v_-, w_+[$ and the horosphere centered at w_+ through x_0, by an easy computation), hence $ds = dt$.



By Equation (52) and Equation (47), we therefore have

$$I_\varphi = \int_{w' \in W^u(v)} \int_{w \in W^{ss}(w')} \varphi(w)\ e^{C_{F,\, w'_+}(\pi(w'), \pi(w))}\ d\mu^\iota_{W^{ss}(w')}(w)\ d\mu_{W^u(v)}(w')\ .$$

This is the first claim of Proposition 3.17. The second one is proven similarly. $\square$

4 Critical exponent and Gurevich pressure

Let $(\widetilde{M}, \Gamma, \widetilde{F})$ be as in the beginning of Chapter 2: $\widetilde{M}$ is a complete simply connected Riemannian manifold, with dimension at least 2 and pinched sectional curvature at most -1; Γ is a nonelementary discrete group of isometries of $\widetilde{M}$; and $\widetilde{F} : T^1\widetilde{M} \to \mathbb{R}$ is a Hölder-continuous Γ-invariant map. The aim of this chapter is to prove results on the critical exponent $\delta_{\Gamma,\, F}$ which are valid under these general hypotheses (in particular, Γ is only assumed to be nonelementary).

4.1 Counting orbit points and periodic geodesics

One way to study the repartition of an orbit of Γ in $\widetilde{M}$ is to count the number of its elements in relatively compact subsets of $\widetilde{M}$. We will count them with weights given by the potential $\widetilde{F}$.

For every $s \geq 0$, for all $x, y \in \widetilde{M}$ and for all open subsets U and V of $\partial_\infty \widetilde{M}$, let

$$G_{\Gamma,\, F,\, x,\, y,\, U,\, V}(t) = \sum_{\gamma \in \Gamma\,:\, d(x, \gamma y) \leq t\,,\ \gamma y \in \mathscr{C}_x U\,,\ \gamma^{-1} x \in \mathscr{C}_y V} e^{\int_x^{\gamma y} \widetilde{F}}\ .$$

The map $s \mapsto G_{\Gamma,\,F,x,\,y,\,U,\,V}(s)$ will be called the *bisectorial orbital counting function* of (Γ, F, U, V). When $V = \partial_\infty \widetilde{M}$, we denote it by $s \mapsto G_{\Gamma,\,F,\,x,\,y,\,U}(s)$ and call it the *sectorial orbital counting function* of (Γ, F, U). When $U = V = \partial_\infty \widetilde{M}$, we denote it by $s \mapsto G_{\Gamma,\,F,\,x,\,y}(s)$ and call it the *orbital counting function* of (Γ, F). When $F = 0$, we recover the usual orbital counting functions of Γ, see for instance [Marg, Rob1] and [Bab3] as well as its references.

Other interesting counting functions are the ones counting periodic orbits of the geodesic flow on T^1M, again with weights given by the potential.

For every periodic (not necessarily primitive) orbit g of length $\ell(g)$ of the geodesic flow $(\phi_t)_{t\in\mathbb{R}}$ on T^1M, let $\mathscr{L}_g$ be the *Lebesgue measure along* g, that is, the measure on T^1M with support g, such that, for every continuous map $f : T^1M \to \mathbb{R}$ and for any $v \in g$,

$$\mathscr{L}_g(f) = \int_0^{\ell(g)} f(\phi_t v)\, dt\ .$$

The *period of g for the potential F* is (for any $v \in g$)

$$\int_g F = \mathscr{L}_g(F) = \int_0^{\ell(g)} F(\phi_t v)\, dt\ .$$

One needs to be careful with the terminology and not confuse the length $\ell(g) = \mathscr{L}_g(1)$ of g and its period $\int_g F = \mathscr{L}_g(F)$ (which depends on F).

For every $s \geq 0$, we denote by $\mathscr{P}\mathrm{er}(s) = \mathscr{P}\mathrm{er}_{T^1M}(s)$ the set of periodic orbits of the geodesic flow in T^1M with length at most s; we allow nonprimitive ones, and, since T^1M is an orbifold, we consider them with multiplicity (the cardinal of the stabiliser in Γ of any $v \in T^1\widetilde{M}$ mapping to an element of g by the canonical map $T^1\widetilde{M} \to T^1M$). Note that only primitive periodic orbits are considered in [Rob1, §5], but when the potential is nonconstant, such a restriction would make our study more difficult.

For every relatively compact open subset W of T^1M, define the *period counting function* of (Γ, F, W) as

$$s \mapsto Z_{\Gamma,\,F,\,W}(s) = \sum_{g\in\mathscr{P}\mathrm{er}(s),\ g\cap W\neq\emptyset} e^{\int_g F}\ .$$

Since M is not assumed to be compact, it is important to assume that W is relatively compact in T^1M, since for instance when M is an infinite Riemannian cover of a compact negatively curved Riemannian manifold, then T^1M contains infinitely many periodic orbits of the same period and of length at most s, for every s large enough. Conversely, if Γ is geometrically finite (see the definition of geometrically finite discrete groups of isometries in the Subsection 8.2), there exists an open relatively compact subset of T^1M meeting every periodic orbit. Note that $Z_{\Gamma,\,F,\,W}(s)$ is nonzero for s large enough if and only if W meets the (topological) nonwandering set $\Omega\Gamma$ of the geodesic flow in T^1M.

We define the *Gurevich pressure* of (Γ, F) as

$$P_{Gur}(\Gamma,\ F) = \limsup_{s\to+\infty}\ \frac{1}{s} \log Z_{\Gamma,\,F,\,W}(s)\ ,$$

where W is any relatively compact open subset of T^1M meeting $\Omega\Gamma$. We will prove in Subsection 4.3 that the Gurevich pressure does not depend on W and that the above upper limit is a limit.

The Gurevich pressure is a (logarithmic) asymptotic growth rate of the periodic orbits of the geodesic flow, weighted by the potential. For Markov shifts on countable alphabets, it has been introduced by Gurevich [Gur1, Gur2] when the potential vanishes, and by Sarig [Sar1, Sar3, Sar2] in general. These last two works have motivated our study. To our knowledge, the Gurevich pressure had not been studied in a non symbolic noncompact context.

We will give in Chapter 9 precise asymptotic results as t goes to $+\infty$ of these counting functions, under stronger hypotheses on $(\widetilde{M}, \Gamma, \widetilde{F})$. In the next subsection, we will only give weaker (logarithmic) asymptotic results, valid in general.

For instance, the following (very weak) result is an easy consequence of Mohsen's shadow lemma 3.10.

Corollary 4.1 *If $\delta_{\Gamma,\,F} > 0$, we have $G_{\Gamma,\,F,\,x,\,y,\,U,\,V}(t) = O(e^{t\,\delta_{\Gamma,\,F}})$ as t goes to $+\infty$.*

Proof. Since $G_{\Gamma,\,F,\,x,\,y,\,U,\,V} \leq G_{\Gamma,\,F,\,x,\,y}$, we may assume that $U = V = \partial_\infty \widetilde{M}$. We may also assume that $\delta_{\Gamma,\,F} < +\infty$. Let $(\mu_x)_{x \in \widetilde{M}}$ be a Patterson density of dimension $\sigma = \delta_{\Gamma,\,F}$ (see Proposition 3.9) for (Γ, F). Then the result follows from Corollary 3.11 (1). □

4.2 Logarithmic growth of the orbital counting functions

In this subsection, we give logarithmic counting results which, though less precise than the results in Subsection 9.2, are valid in a much greater generality (they only require Γ to be nonelementary).

The first result shows that the upper limit defining the critical exponent $\delta_{\Gamma,\,F}$ of (Γ, F) is in fact a limit. Its proof follows closely the proof of the main result of [Rob2] (corresponding to the case $F = 0$).

Theorem 4.2 *Let $\widetilde{M}$ be a complete simply connected Riemannian manifold, with dimension at least 2 and pinched sectional curvature at most -1, and $x, y \in \widetilde{M}$. Let Γ be a nonelementary discrete group of isometries of $\widetilde{M}$. Let $\widetilde{F} : T^1\widetilde{M} \to \mathbb{R}$ be a Hölder-continuous Γ-invariant map. Then*

$$\delta_{\Gamma,\,F} = \lim_{n \to +\infty} \frac{1}{n} \log \sum_{\gamma \in \Gamma,\; d(x, \gamma y) \leq n} e^{\int_x^{\gamma y} \widetilde{F}} \;.$$

With the previous notation, this can be written as

$$\lim_{s \to +\infty} \frac{1}{s} \log\; G_{\Gamma,\,F,\,x,\,y}(s) = \delta_{\Gamma,\,F} \;,$$

that is, the orbital counting function grows logarithmically as $s \mapsto e^{\delta_{\Gamma,\,F}\, s}$. We will prove in Corollary 4.3 that the sectorial and bisectorial counting functions grow similarly (under obvious conditions).

Proof. Let $\delta = \delta_{\Gamma,\,F} > -\infty$ (by Lemma 3.3 (v)). For every $x' \in \widetilde{M}$, with $a_n = \sum_{\gamma \in \Gamma,\; d(x', \gamma y) \leq n} e^{\int_{x'}^{\gamma y} \widetilde{F}}$, we have seen in Equation (17) that $\delta = \limsup_{n \to +\infty} \frac{1}{n} \log a_n$.

For a contradiction, assume that $\liminf_{n\to+\infty} \frac{1}{n}\log a_n < \delta$. Note that this lower limit does not depend on x'. Hence there exist a sequence $(n_k)_{k\in\mathbb{N}}$ of positive integers and $\sigma \in]-\infty, \delta[$ such that for every $x' \in \widetilde{M}$, for every $k \in \mathbb{N}$ big enough,

$$a_{n_k} \leq e^{\sigma n_k} \ . \tag{57}$$

Let us construct a Patterson density of dimension σ for (Γ, F), which contradicts Corollary 3.11 (2) since $\sigma < \delta$.

Let $\mathscr{D}_z$ be the (unit) Dirac mass at a point $z \in \widetilde{M}$. For all $t \in [0, +\infty[$ and $x' \in \widetilde{M}$, let

$$\nu_{x',\,t} = \frac{\sum_{\gamma\in\Gamma,\ d(x',\,\gamma y)\leq t} \ e^{\int_{x'}^{\gamma y}(\widetilde{F}-\sigma)}\ \mathscr{D}_{\gamma y}}{\sum_{\gamma\in\Gamma,\ d(y,\,\gamma y)\leq t} \ e^{\int_{y}^{\gamma y}(\widetilde{F}-\sigma)}} \ .$$

By compactness, for every $r \geq 0$, there exists a sequence $\big(k_i(r)\big)_{i\in\mathbb{N}}$ of positive integers such that the probability measures $\nu_{y,\,n_{k_i(r)}-r}$ weak star converge to a probability measure $\mu_{y,\,r}$. Since the Poincaré series $\sum_{\gamma\in\Gamma} \ e^{\int_{y}^{\gamma y}(\widetilde{F}-\sigma)}$ diverges (as $\sigma < \delta$), the support of the measure $\mu_{y,\,r}$ is $\Lambda\Gamma$. For every $x' \in \widetilde{M}$, define

$$d\mu_{x',\,r}(\xi) = e^{C_{F-\sigma,\,\xi}(x',y)}\ d\mu_{y,\,r}(\xi) \ ,$$

and let us prove that $\nu_{x',\,n_{k_i(r)}-r}$ weak star converges to $\mu_{x',\,r}$ as $i \to +\infty$ if $r \geq d(x', y)$.

Note that if $r \geq d(x', y)$, for every $k \in \mathbb{N}$ large enough, by the triangle inequality and Equation (57), we have

$$\Big\| \sum_{\gamma\in\Gamma,\ d(x',\gamma y)\leq n_k - r} e^{\int_{x'}^{\gamma y}(\widetilde{F}-\sigma)}\ \mathscr{D}_{\gamma y} - \sum_{\gamma\in\Gamma,\ d(y,\gamma y)\leq n_k - r} e^{\int_{x'}^{\gamma y}(\widetilde{F}-\sigma)}\ \mathscr{D}_{\gamma y} \Big\|$$
$$\leq \sum_{\gamma\in\Gamma,\ n_k - r - d(x',y)\leq d(x',\gamma y)\leq n_k - r + d(x',y)} e^{\int_{x'}^{\gamma y}(\widetilde{F}-\sigma)} \leq\ e^{(2r-n_k)\sigma}\ a_{n_k} \leq e^{2r\sigma} \ .$$

Since the denominator of $\nu_{x',\,n_{k_i(r)}-r}$ tends to $+\infty$ as $i \to +\infty$ and since

$$\Big| \int_{x'}^{\gamma y} (\widetilde{F}-\sigma) - \int_{y}^{\gamma y} (\widetilde{F}-\sigma) - C_{F-\sigma,\,\xi}(x', y) \Big|$$

is arbitrarily small if γy is close enough to $\xi \in \partial_\infty \widetilde{M}$, we have the convergence we looked for.

Now, for all $\gamma \in \Gamma$, $t \geq 0$ and $x' \in \widetilde{M}$, we clearly have $\gamma_* \nu_{x',\,t} = \nu_{\gamma x',\,t}$. Hence if $r \geq \max\{d(x', y), d(\gamma x', y)\}$, then $\gamma_*\mu_{x',\,r} = \mu_{\gamma x',\,r}$ by taking limits. By compactness, there exists a sequence $(r_j)_{j\in\mathbb{N}}$ in $[0, +\infty[$ converging to $+\infty$ such that the probability measures $\nu_{y,\,r_j}$ weak star converge to a probability measure μ_y. If $(\mu_{x'})_{x'\in\widetilde{M}}$ is the family of measures on $\partial_\infty \widetilde{M}$ defined by

$$d\mu_{x'}(\xi) = e^{C_{F-\sigma,\,\xi}(x',y)}\ d\mu_y(\xi) \ ,$$

the sequence $(\mu_{x',\,r_j})_{j\in\mathbb{N}}$ weak star converges to $\mu_{x'}$, and hence $(\mu_{x'})_{x'\in\widetilde{M}}$ is a Patterson density of dimension σ for (Γ, F), a contradiction. □

Remark. Here is a short proof of this result, using the subadditivity ideas of [DaPS]. This paper proves that

• there exist a finite subset P of Γ and $c \geq 0$ such that for all $\alpha, \beta \in \Gamma$, there exists $\gamma \in P$ such that the Hausdorff distance between the geodesic segment $[x, \alpha\gamma\beta y]$ and $[x, \alpha y] \cup [\alpha y, \alpha\gamma x] \cup [\alpha\gamma x, \alpha\gamma\beta y]$ is at most c;

• let $(b_n)_{n\in\mathbb{N}}$ be a sequence of nonnegative real numbers such that there exist $c > 0$ and $N \in \mathbb{N}$ such that for all $n, m \in \mathbb{N}$, we have

$$b_n b_m \leq c \sum_{i=-N}^{N} b_{n+m+i} ,$$

then, with $a_n = \sum_{k=0}^{n-1} b_n$, the limit of $a_n^{\frac{1}{n}}$ as $n \to +\infty$ exists.

The first point implies that there exists $c' \geq 0$ (depending on x, y) such that for all $\alpha, \beta \in \Gamma$, there exists $\gamma \in P$ satisfying

$$\big|\, d(x, \alpha\gamma\beta y) - d(x, \alpha y) - d(x, \beta y) \,\big| \leq c'$$

and, since $\widetilde{F}$ is Hölder-continuous and bounded on compact subsets of $\widetilde{M}$,

$$\Big| \int_x^{\alpha\gamma\beta y} \widetilde{F} - \int_x^{\alpha y} \widetilde{F} - \int_x^{\beta y} \widetilde{F} \Big| \leq c' .$$

Hence the sequence $\Big(b_n = \sum_{\gamma\in\Gamma,\, n\leq d(x,\gamma y)<n+1} e^{\int_x^{\gamma y} \widetilde{F}}\Big)_{n\in\mathbb{N}}$ satisfies the assumptions of the second point, and Theorem 4.2 follows.

Corollary 4.3 *With the assumptions of Theorem 4.2,*

(1) if U is an open subset of $\partial_\infty \widetilde{M}$ meeting $\Lambda\Gamma$, then $\lim_{t\to+\infty} \frac{1}{t} \log G_{\Gamma,\,F,\,x,\,y,\,U}(t) = \delta_{\Gamma,\,F}$;

(2) if U and V are any two open subsets of $\partial_\infty \widetilde{M}$ meeting the limit set $\Lambda\Gamma$, then $\lim_{t\to+\infty} \frac{1}{t} \log G_{\Gamma,\,F,\,x,\,y,\,U,\,V}(t) = \delta_{\Gamma,\,F}$.

Proof. We adapt the proof of [Rob2, Coro. 1] which corresponds to the case $F = 0$. The validity of these statements for all open sets U and V is independent of x and y, since for all $z, z' \in \widetilde{M}$ and all open subsets W, W' of $\partial_\infty \widetilde{M}$ such that $\overline{W} \subset W'$, there exists a compact subset K of $\widetilde{M}$ such that $\mathscr{C}_z W \subset K \cup \mathscr{C}_{z'} W'$, and since the intersection of K with any orbit of Γ is a finite. Recall that the convex hull $\mathscr{C}\Lambda\Gamma$ of $\Lambda\Gamma$ is Γ-invariant. We may assume that $y \in \mathscr{C}\Lambda\Gamma$, hence that the orbit of y is contained in $\mathscr{C}\Lambda\Gamma$.

(1) For every $t \in [0, +\infty[$, for every open subset U' of $\widetilde{M} \cup \partial_\infty \widetilde{M}$ and for every $z \in \widetilde{M}$, let

$$a_{t,\,U',\,z} = \sum_{\gamma\in\Gamma \,:\, d(z,\gamma y)\leq t\,,\; \gamma y\in U'} e^{\int_z^{\gamma y} \widetilde{F}} .$$

Note that for every $\gamma \in \Gamma$, by Equation (13) and a change of variable in this sum, we have

$$a_{t,\,U',\,z} = a_{t,\,\gamma U',\,\gamma z} . \tag{58}$$

When U' is the open set $\mathscr{C}_x U - \{x\}$, the difference $G_{\Gamma,\,F,\,x,\,y,\,U}(t) - a_{t,\,U',\,x}$ is equal to 1 if x belongs to the Γ-orbit of y, and 0 otherwise. Since $G_{\Gamma,\,F,\,x,\,y,\,U}(t) \leq G_{\Gamma,\,F,\,x,\,y}(t)$ and by Theorem 4.2, we have

$$\limsup_{t\to+\infty} \frac{1}{t} \log G_{\Gamma,\,F,\,x,\,y,\,U}(t) \leq \delta_{\Gamma,\,F} .$$

Hence to prove the first assertion of Corollary 4.3, we only have to prove that

$$\liminf_{t\to+\infty}\frac{1}{t}\log a_{t,\,U',\,x}\geq \delta_{\Gamma,\,F}$$

for every open subset U' of $\widetilde{M}\cup\partial_\infty\widetilde{M}$ meeting $\Lambda\Gamma$.

Let U' be such an open subset. Since Γ is nonelementary, every orbit in $\Lambda\Gamma$ is dense. Hence by compactness of $\Lambda\Gamma$, there exist $\gamma_1,\dots,\gamma_k$ in Γ such that $\Lambda\Gamma\subset\bigcup_{i=1}^k\gamma_iU'$. In particular $\mathscr{C}\Lambda\Gamma-\bigcup_{i=1}^k\gamma_iU'$ is compact. Hence there exists $c_1'\geq 0$ such that for every $t\in[0,+\infty[\,$, we have

$$G_{\Gamma,\,F,\,x,\,y}(t)\leq c_1'+\sum_{i=1}^k\ a_{t,\,\gamma_iU',\,x}\ . \tag{59}$$

Let $r=\max_{1\leq i\leq k}\ d(x,\gamma_i x)$ and $t\in[r,+\infty[\,$. Note that for all $i\in\{1,\dots,k\}$ and $\gamma\in\Gamma$, if $d(\gamma_i^{-1}x,\gamma y)\leq t-r$, then $d(x,\gamma y)\leq t$. Furthermore, by Lemma 3.2, there exists $c_2'>0$ such that for all $i\in\{1,\dots,k\}$ and $\gamma\in\Gamma$, we have

$$\Big|\int_x^{\gamma y}\widetilde{F}-\int_{\gamma_i^{-1}x}^{\gamma y}\widetilde{F}\Big|\leq c_2'\ .$$

Hence, for all $i\in\{1,\dots,k\}$ and $t\in[r,+\infty[\,$, we have, using Equation (58),

$$a_{t-r,\,\gamma_iU',\,x}=a_{t-r,\,U',\,\gamma_i^{-1}x}\leq e^{c_2'}\ a_{t,\,U',\,x}\ .$$

Therefore, by Equation (59), we have

$$a_{t,\,U',\,x}\geq\frac{1}{k\ e^{c_2'}}\big(G_{\Gamma,\,F,\,x,\,y}(t-r)-c_1'\big)\ . \tag{60}$$

By taking the logarithm, dividing by t and taking the lower limit as $t\to+\infty$, the result then follows from Theorem 4.2.

(2) The proof of the second assertion reduces to the first one, using a similar approach as the reduction from the first one to Theorem 4.2. For every $t\in[0,+\infty[\,$, for all open subsets U',V' of $\widetilde{M}\cup\partial_\infty\widetilde{M}$ and for all $z,w\in\widetilde{M}$, we now introduce

$$b_{t,\,U',\,V',\,z,\,w}=\sum_{\gamma\in\Gamma\,:\,d(z,\gamma w)\leq t\,,\ \gamma w\in U',\ \gamma^{-1}z\in V'} e^{\int_z^{\gamma w}\widetilde{F}}\ , \tag{61}$$

which satisfies $b_{t,\,U',\,V',\,z,\,w}=b_{t,\,U',\,\alpha V',\,z,\,\alpha w}$ for every $\alpha\in\Gamma$. As above, we only have to prove that

$$\liminf_{t\to+\infty}\frac{1}{t}\log b_{t,\,U',\,V',\,x,\,y}\geq\delta_{\Gamma,\,F} \tag{62}$$

for all open subsets U',V' of $\widetilde{M}\cup\partial_\infty\widetilde{M}$ meeting $\Lambda\Gamma$.

Let U',V' be two such open subsets. As above, there exist $\alpha_1,\dots,\alpha_\ell$ in Γ such that $\Lambda\Gamma\subset\bigcup_{i=1}^\ell\alpha_iV'$, and hence there exists $c_3'\geq 0$ such that for every $t\in[0,+\infty[\,$, we have

$$a_{t,\,U',\,x}\leq c_1'+\sum_{i=1}^\ell\ b_{t,\,U',\,\alpha_iV',\,x,\,y}\ .$$

Let now $r = \max_{1\le i\le \ell}\; d(y, \alpha_i y)$. For all $i \in \{1, \dots, \ell\}$, $t \in [r, +\infty[$ and $\gamma \in \Gamma$, if $d(x, \gamma\alpha_i^{-1} y) \le t - r$, then $d(x, \gamma y) \le t$. Furthermore, by Equation (13) and by Lemma 3.2, there exists $c'_4 > 0$ such that for all $i \in \{1, \dots, \ell\}$ and $\gamma \in \Gamma$, we have

$$\Big|\int_x^{\gamma y} \widetilde{F} - \int_x^{\gamma\alpha_i^{-1} y} \widetilde{F}\Big| = \Big|\int_y^{\gamma^{-1} x} \widetilde{F}\circ\iota - \int_{\alpha_i^{-1} y}^{\gamma^{-1} x} \widetilde{F}\circ\iota\Big| \le c'_4 \;.$$

Hence, for all $i \in \{1, \dots, k\}$ and $t \in [r, +\infty[\,$, we have

$$b_{t-r,\, U',\, \alpha_i V',\, x,\, y} = b_{t-r,\, U',\, V',\, x,\, \alpha_i^{-1} y} \le e^{c'_4}\; b_{t,\, U',\, V',\, x,\, y} \;.$$

Therefore

$$b_{t,\, U',\, V',\, x,\, y} \ge \frac{1}{\ell\; e^{c'_4}} \big(a_{t-r,\, U',\, x} - c'_3\big) \;,$$

and we conclude, using Assertion (1), as in the end of the proof of this assertion. □

4.3 Equality between critical exponent and Gurevich pressure

The aim of this subsection is stated in its title: we now prove that the logarithmic growth rate of the periodic geodesics (weighted by the potential) is equal to the logarithmic growth rate of the orbit points (weighted by the potential).

Theorem 4.4 *Let $\widetilde{M}$ be a complete simply connected Riemannian manifold, with dimension at least 2 and pinched sectional curvature at most -1, and $x, y \in \widetilde{M}$. Let Γ be a nonelementary discrete group of isometries of $\widetilde{M}$. Let $\widetilde{F} : T^1\widetilde{M} \to \mathbb{R}$ be a Hölder-continuous Γ-invariant map. Let W be a relatively compact open subset of T^1M meeting the (topological) nonwandering set $\Omega\Gamma$. Then*

$$P_{Gur}(\Gamma, F) = \lim_{s\to+\infty} \frac{1}{s} \log Z_{\Gamma,\, F,\, W}(s) = \delta_{\Gamma,\, F} \;.$$

This proves in particular that the Gurevich pressure does not depend on W and that the upper limit defining it is a limit.

Proof. Let $\mathrm{pr} : T^1\widetilde{M} \to T^1M = \Gamma\backslash T^1\widetilde{M}$ be the canonical projection.

Let us first prove that

$$\limsup_{s\to+\infty} \frac{1}{s} \log Z_{\Gamma,\, F,\, W}(s) \le \delta_{\Gamma,\, F} \;.$$

Since W is relatively compact, there exists a compact subset K of $\widetilde{M}$ such that $\mathrm{pr}(\pi^{-1}(K))$ contains W. Let r be the diameter of K, and fix $x \in K$.

For all $s \ge 0$ and g in $\mathscr{P}\mathrm{er}(s)$ such that $g \cap W \ne \emptyset$, let γ_g be one of the hyperbolic elements of Γ whose translation axis Axe_{γ_g} meets K, whose translation length is the length $\ell(g)$ of g and such that for every $y \in Axe_{\gamma_g}$, the image by pr of the unit tangent vector at y pointing towards $\gamma_g y$ belongs to g. Note that the number of these elements γ_g is at least equal to the cardinality of the pointwise stabiliser of Axe_{γ_g}, that is to the multiplicity of g.

Let x_g be the closest point to x on Axe_{γ_g}, which satisfies $d(x, x_g) \le r$ since $x \in K$ and Axe_{γ_g} meets K. We have by the triangle inequality

$$d(x, \gamma_g x) \le d(x, x_g) + d(x_g, \gamma_g x_g) + d(\gamma_g x, \gamma_g x_g) \le \ell(g) + 2r \le s + 2r \;.$$

Furthermore, by (two applications of) Lemma 3.2 with $r_0 = r$, there exists a constant $c \geq 0$ (depending only on r, the Hölder constants of $\widetilde{F}$, the bounds on the sectional curvature and $\max_{\pi^{-1}(B(x,r))} |\widetilde{F}|$) such that

$$\Big| \int_x^{\gamma_g x} \widetilde{F} - \int_g \widetilde{F} \Big| = \Big| \int_x^{\gamma_g x} \widetilde{F} - \int_{x_g}^{\gamma_g x_g} \widetilde{F} \Big| \leq c\,.$$

Hence $Z_{\Gamma,\,F,\,W}(s) \leq e^c\, G_{\Gamma,\,F,\,x,\,x}(s+2r)$, which proves our first claim, by Theorem 4.2.

In order to prove Theorem 4.4, we now only have to prove that

$$\liminf_{s\to+\infty} \frac{1}{s} \log Z_{\Gamma,\,F,\,W}(s) \geq \delta_{\Gamma,\,F}\,.$$

Let $v \in T^1\widetilde{M}$ such that $\mathrm{pr}(v) \in W \cap \Omega\Gamma$, and now let $x = \pi(v)$. Note that the points v_- and v_+ both belong to $\Lambda\Gamma$.

By standard arguments (see for instance Lemma 2.7 and [GdlH, page 150-151]), there exist U' and V' small enough neighbourhoods in $\widetilde{M} \cup \partial_\infty \widetilde{M}$ of v_+ and v_- respectively, such that for every $\gamma \in \Gamma$ such that $\gamma x \in U'$ and $\gamma^{-1}x \in V'$, then γ is a hyperbolic element and v is close to its translation axis Axe_γ, in the sense that the point x is at distance at most 1 from some point x_γ in Axe_γ, and that if v_γ is the unit tangent vector at x_γ pointing towards γx_γ, then $\mathrm{pr}(v_\gamma) \in W$ (recall that W is open). Note that U' and V' meet $\Lambda\Gamma$.

Let $s \geq 0$ and let $\gamma \in \Gamma$ be such that $d(x,\gamma x) \leq s$, $\gamma x \in U'$ and $\gamma^{-1}x \in V'$. Note that the orbit g_γ under the geodesic flow of $\mathrm{pr}(v_\gamma)$ is periodic, of length

$$\ell(g_\gamma) \leq d(x,\gamma x) \leq s\,,$$

and as above, there exists $c \geq 0$ such that

$$\Big| \int_{g_\gamma} \widetilde{F} - \int_x^{\gamma x} \widetilde{F} \Big| = \Big| \int_{x_\gamma}^{\gamma x_\gamma} \widetilde{F} - \int_x^{\gamma x} \widetilde{F} \Big| \leq c\,.$$

Hence $Z_{\Gamma,\,F,\,W}(s) \geq e^{-c}\, b_{s,\,U',\,V',\,x,\,x}$ with the notation of Equation (61), which proves our second claim, by Equation (62). □

The next result is an obvious corollary of Theorem 4.4, also following from Remark 3.1 and the remark at the end of Subsection 3.2.

Corollary 4.5 *Under the assumption of the previous theorem, if $\widetilde{G} : T^1\widetilde{M} \to \mathbb{R}$ is another Hölder-continuous Γ-invariant map, such that the periods of every $\gamma \in \Gamma$ for $\widetilde{F}$ and $\widetilde{G}$ are the same, then the critical exponents of (Γ, F) and (Γ, G) are the same :*

$$\delta_{\Gamma,\,F} = \delta_{\Gamma,\,G}\,.$$

4.4 Critical exponent of Schottky semigroups

The main aim of this subsection is to prove (see Theorem 4.8) that the critical exponent $\delta_{\Gamma,\,F}$ is the upper bound of the critical exponents $\delta_{G,\,F}$ (defined as for groups) of finitely generated subsemigroups G of Γ, which have strong geometric properties (of Schottky type). The constructions of this subsection will also be useful for the proof of the Variational principle in Subsection 6.2.

Let $\widetilde{M}$ be a complete simply connected Riemannian manifold, with dimension at least 2 and pinched negative sectional curvature at most -1. Let G be a discrete semigroup of isometries of $\widetilde{M}$. We refer to [Mer] for general information about discrete semigroups of isometries of CAT(-1)-spaces.

Let $\widetilde{F} : T^1\widetilde{M} \to \mathbb{R}$ be a Hölder-continuous G-invariant map. We define the *limit set* ΛG of G (as the set of accumulation points in $\partial_\infty \widetilde{M}$ of any orbit Gx_0), the *Poincaré series*

$$Q_{G,\,F,\,x,\,y}(s) = \sum_{\gamma \in G} \; e^{\int_x^{\gamma y} (\widetilde{F}-s)}$$

of (G, F), and the *critical exponent*

$$\delta_{G,\,F} = \limsup_{n\to+\infty} \frac{1}{n} \log \sum_{\gamma\in G,\; n-1<d(x,\gamma y)\leq n} e^{\int_x^{\gamma y} \widetilde{F}}$$

of (G, F), exactly as for the case of groups.

Recall that a map $f : X \to Y$ between metric spaces is *quasi-isometric* if there exist $\lambda \geq 1$ and $c \geq 0$ such that for all $x, y \in X$, we have

$$-c + \frac{1}{\lambda}\, d(x,y) \leq d(f(x), f(y)) \leq \lambda\, d(x,y) + c\,.$$

If G is generated (as a semigroup) by a finite set S, and is endowed with the (semigroup) word metric defined by S (see below when G is free on S), we say that G is *convex-cocompact* if the map from G to $\widetilde{M}$ defined by $g \mapsto gx_0$ is quasi-isometric (for any $x_0 \in \widetilde{M}$).

We first give a construction of convex-cocompact free semigroups of isometries of $\widetilde{M}$ (of Schottky type). We have not tried to get optimal constants.

Proposition 4.6 *For every $\epsilon \in \,]0, 1]$, there exists $\theta > 0$ such that for every set S of isometries of $\widetilde{M}$, for all $N \geq 5$ and $C \geq 6$, for all $x_0 \in \widetilde{M}$ and $v_0 \in T^1_{x_0}\widetilde{M}$ such that the following conditions are satisfied:*

(1) $N \leq d(x_0, \alpha x_0) < N + 1$ for every $\alpha \in S$,

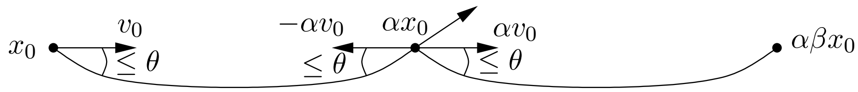
(2) $d(\alpha x_0, \beta x_0) \geq C$ for all distinct α and β in S,

(3) for every $\alpha \in S$, the angle at x_0 between v_0 and the unit tangent vector at x_0 pointing towards αx_0, and the angle at αx_0 between $-\alpha v_0$ and the unit tangent vector at αx_0 pointing towards x_0, are at most θ (see the picture below),

then the semigroup G generated by S is free on S and convex-cocompact, and for all γ, γ' in G, the piecewise geodesic segment $[x_0, \gamma x_0] \cup [\gamma x_0, \gamma\gamma' x_0]$ is contained in the ϵ-neighbourhood of $[x_0, \gamma\gamma' x_0]$. Furthermore

$$k(N - 2\epsilon) \leq d(x_0, \gamma x_0) \leq k(N + 1)$$

for every word γ in the elements of S of length k. Lastly, for all distinct words γ and γ' in the elements of S with the same length, there exist $x \in [x_0, \gamma x_0]$ and $x' \in [x_0, \gamma' x_0]$ such that $d(x_0, x) = d(x_0, x')$ and $d(x, x') \geq 1 - 8\epsilon$.



Proof. Fix $\epsilon \in \,]0,1]$, and let $\theta = \min\{\frac{\pi}{4}, \frac{1}{2}\theta_0(5, \epsilon, \frac{\pi}{4})\}$, where $\theta_0(\cdot,\cdot,\cdot)$ has been defined in Lemma 2.6. Let S, N, C, x_0, v_0 be as in the statement. Note that the assumptions (1) and (2) imply that S is finite. Before proving that G is free and convex-cocompact, we start with a few preliminary remarks.

For every word $m = \alpha_1 \dots \alpha_k$ in elements of S, we denote by $\ell(m) = k \in \mathbb{N}$ its length (the only word with zero length being the empty word) and, for $0 \leq j \leq k$, by $m_j = \alpha_1 \dots \alpha_j$ its initial subword of length j (in particular m_0 is the empty word). If $m = \alpha_1 \dots \alpha_k$ and $m' = \beta_1 \dots \beta_{k'}$ are two words in elements of S, let

$$j(m, m') = \max\{i \in \mathbb{N} \;:\; \forall\, i' \leq i, \;\; \alpha_{i'} = \beta_{i'}\}$$

be the length of the maximal common initial subword of m and m'. It is 0 if and only if m or m' is the empty word or $\alpha_1 \neq \beta_1$. Recall that the distance between the words m and m' is

$$d(m, m') = \ell(m) + \ell(m') - 2j(m, m') \;.$$

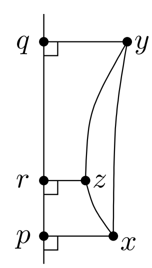
Consider the piecewise geodesic path

$$\omega_m = [m_0 x_0, m_1 x_0] \cup [m_1 x_0, m_2 x_0] \cup \dots \cup [m_{k-1} x_0, m_k x_0]$$

(with $\omega_m = \{x_0\}$ by convention if m is the empty word). By Assumption (3), the exterior angles of ω_m at the points $m_i x_0$ for $1 \leq i \leq k-1$ are at most 2θ (see the picture below). By Assumption (1), the length of each segment of ω_m is at least $N \geq 5$. Hence, by Lemma 2.6 and the definition of θ, the piecewise geodesic path ω_m is contained in the ϵ-neighbourhood of $[x_0, m x_0]$. In particular, this proves the penultimate assertion of Proposition 4.6, by convexity.

Since $N \geq 5$, $\epsilon \leq 1$ and $2\theta \leq \frac{\pi}{2}$, the closest points to $m_i x_0$ on $[x_0, m x_0]$ for $0 \leq i \leq k$ are in this order on this segment.

Indeed, if x, y, z are points in $\widetilde{M}$ such that $N \leq d(x,y) \leq N+1$, $N \leq d(y,z) \leq N+1$, $\measuredangle_y(x,z) \geq \frac{\pi}{2}$, having closest points respectively p, q, r at distance at most ϵ on a geodesic segment, with the absurd hypothesis that $r \in [p,q]$, then by the triangle inequality and since closest point maps do not increase distances, we have



$$\begin{aligned} d(x,z) &\leq d(p,r) + 2\epsilon = d(p,q) - d(r,q) + 2\epsilon \leq d(x,y) - d(z,y) + 4\epsilon \\ &\leq N + 1 - N + 4\epsilon = 1 + 4\epsilon \leq 5 \leq N \;. \end{aligned}$$

By an angle comparison, the angle $\measuredangle_y(x,z)$ is less than $\frac{\pi}{2}$, a contradiction. [Another argument is that since $\measuredangle_y(x,z) \geq \frac{\pi}{2}$, we have by a distance comparison that $d(x,z) \geq d(x,y) + d(y,z) - 2\log(1+\sqrt{2}) \geq 2N - 2\log(1+\sqrt{2}) > N$, a contradiction.]

We now give our last preliminary remark, which yields the penultimate assertion of Proposition 4.6. By the triangle inequality and Assumption (1), we have

$$d(m x_0, x_0) \leq \sum_{i=0}^{k-1} d(m_i x_0, m_{i+1} x_0) \leq k(N+1) = (N+1)\,\ell(m) \;,$$

on one hand, and on the other hand, with p_i the closest point to $m_i x_0$ on $[x_0, mx_0]$ for $0 \leq i \leq k$, using the above ordering remark,

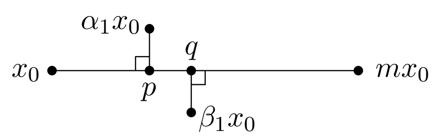
$$d(mx_0, x_0) = \sum_{i=0}^{k-1} d(p_i, p_{i+1}) \geq \sum_{i=0}^{k-1} (d(m_i x_0, m_{i+1} x_0) - 2\epsilon)$$
$$\geq k(N - 2\epsilon) = (N - 2\epsilon)\, \ell(m)\ .$$

Note that $N - 2\epsilon > 0$ by the assumptions on N and ϵ. In particular, if m is not the empty word, then $mx_0 \neq x_0$.

After these preliminary remarks, let us prove that G is free on S. Let $m = \alpha_1 \dots \alpha_k$ and $m' = \beta_1 \dots \beta_{k'}$ be two distinct words in elements of S, and let us prove that $mx_0 \neq m'x_0$, which yields the result.

Assume for a contradiction that $mx_0 = m'x_0$. Let $j = j(m, m')$. Up to multiplying m and m' by the inverse of their maximal common initial subword, we may assume that $j = 0$. We have $\min\{k, k'\} > 0$, otherwise, up to exchanging m and m', we have $k \neq 0$ and $k' = 0$, so that $mx_0 = x_0$ and m is not the empty word, a contradiction to the last preliminary remark. Hence α_1 and β_1 exist and are distinct.

Let p and q be the closest points to $\alpha_1 x_0$ and $\beta_1 x_0$ on $[x_0, mx_0]$. Up to permuting m and m', we may assume that x_0, p, q, mx_0 are in this order on $[x_0, mx_0]$.



By the triangle inequality, since closest point maps do not increase distances, by Assumption (1) and by the properties of ϵ and C, we have

$$d(\alpha_1 x_0, \beta_1 x_0) \leq d(p, q) + 2\epsilon = d(q, x_0) - d(p, x_0) + 2\epsilon$$
$$\leq d(\beta_1 x_0, x_0) - d(\alpha_1 x_0, x_0) + 3\epsilon \leq N + 1 - N + 3\epsilon = 3\epsilon + 1 < C\ .$$

This contradicts Assumption (2). Hence G is indeed free on S.

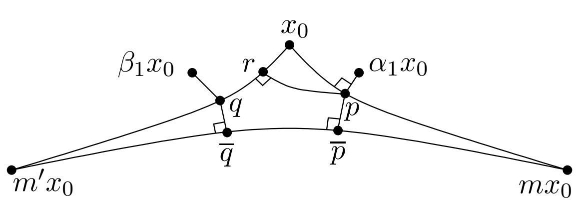
Finally, let us prove that the map from G to $\widetilde{M}$ defined by $g \mapsto gx_0$ is quasi-isometric, which says that G is convex-cocompact. Let $m = \alpha_1 \dots \alpha_k$ and $m' = \beta_1 \dots \beta_{k'}$ be two words in elements of S. Let us prove that

$$(N - 2\epsilon) d(m, m') - 2(\log(1 + \sqrt{2}) + N - \epsilon) \leq d(mx_0, m'x_0) \leq (N + 1)\, d(m, m')\ , \quad (63)$$

which implies the result.

Let $j = j(m, m')$. Up to multiplying m and m' by the inverse of their maximal common initial subword, we may assume that $j = 0$. By the last preliminary remark, we may assume that neither m nor m' is the empty word (we would not have to do this in the group case, but G is here only a semigroup). We may also assume that $m \neq m'$. Hence α_1 and β_1 exist and are distinct.

Let p be the closest point to $\alpha_1 x_0$ on $[x_0, mx_0]$. Let r and q be the closest points to p and $\beta_1 x_0$ respectively on $[x_0, m'x_0]$. Let $\overline{p}$ and $\overline{q}$ be the closest points to p and q respectively on $[mx_0, m'x_0]$.



Consider the geodesic triangle in the CAT(-1)-space $\widetilde{M}$ with vertices $mx_0, x_0, m'x_0$. Each of its sides is contained in the $\log(1+\sqrt{2})$-neighbourhood of the union of the two other sides.

Assume for a contradiction that $d(p,r) \leq \log(1+\sqrt{2})$. As when we proved that G is free, we have

$$d(\alpha_1 x_0, \beta_1 x_0) \leq d(r,q) + 2\epsilon + \log(1+\sqrt{2}) = |d(q,x_0) - d(r,x_0)| + 2\epsilon + \log(1+\sqrt{2}) \;.$$

But by Assumption (1), we have $N - \epsilon \leq d(q,x_0) \leq N+1$,

$$d(r,x_0) \leq d(p,x_0) \leq d(\alpha_1 x_0, x_0) \leq N+1$$

and

$$\begin{aligned} d(r,x_0) &\geq d(p,x_0) - \log(1+\sqrt{2}) \geq d(\alpha_1 x_0, x_0) - \epsilon - \log(1+\sqrt{2}) \\ &\geq N - \epsilon - \log(1+\sqrt{2}) \;. \end{aligned}$$

In particular, $|d(q,x_0) - d(r,x_0)| \leq 1 + \epsilon + \log(1+\sqrt{2})$ and, since $\epsilon \leq 1$,

$$d(\alpha_1 x_0, \beta_1 x_0) \leq 1 + 3\epsilon + 2\log(1+\sqrt{2}) < 6 \;.$$

Since $C \geq 6$, this contradicts Assumption (2).

By the above property of the geodesic triangles, we hence have $d(p,\overline{p}) \leq \log(1+\sqrt{2})$. Similarly, $d(q,\overline{q}) \leq \log(1+\sqrt{2})$.

Let us prove the upper bound in Equation (63). Indeed, by the triangle inequality and the last preliminary remark, since $j(m,m') = 0$, we have

$$d(mx_0, m'x_0) \leq d(mx_0, x_0) + d(x_0, m'x_0) \leq (N+1)k + (N+1)k' = (N+1)d(m,m') \;.$$

Let us prove the lower bound in Equation (63). By convexity, the points $mx_0, \overline{p}, \overline{q}, m'x_0$ are in this order on $[mx_0, m'x_0]$. Hence by the triangle inequality, since α_1 is the first letter of m and β_1 is the first letter of m', by the last preliminary remark, and since $j(m,m') = 0$, we have

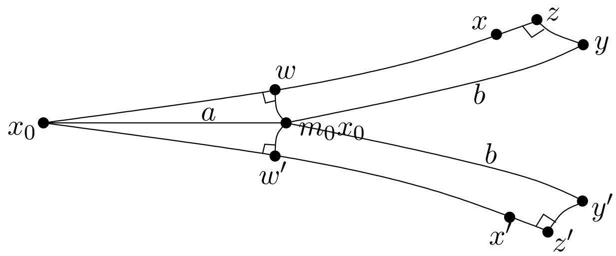
$$\begin{aligned} d(mx_0, m'x_0) &\geq d(mx_0, \overline{p}) + d(\overline{q}, m'x_0) \geq d(mx_0, p) + d(q, m'x_0) - 2\log(1+\sqrt{2}) \\ &\geq d(mx_0, \alpha_1 x_0) + d(\beta_1 x_0, m'x_0) - 2\log(1+\sqrt{2}) - 2\epsilon \\ &\geq (N-2\epsilon)(k-1) + (N-2\epsilon)(k'-1) - 2\log(1+\sqrt{2}) - 2\epsilon \\ &= (N-2\epsilon)d(m,m') - 2(\log(1+\sqrt{2}) + N - \epsilon) \;, \end{aligned}$$

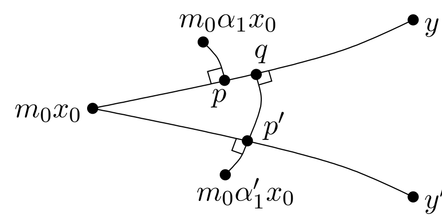
as required.

Let us finally prove the last assertion of Proposition 4.6. Let m and m' be two distinct words in the elements of S with the same length. We may hence write $m = m_0 \alpha_1 m_1$ and $m' = m_0 \alpha'_1 m'_1$ where m_0, m_1, m'_1 are (possibly empty) words in the elements of S with $\ell(m_0) = j(m,m')$ and α_1, α'_1 are distinct elements of S. Let

$$a = d(x_0, m_0 x_0) \quad \text{and} \quad b = \min\{d(m_0 x_0, m x_0),\ d(m_0 x_0, m' x_0)\} \;.$$

Let y and y' be the points of $[m_0 x_0, m x_0]$ and $[m_0 x_0, m' x_0]$ respectively at distance b from $m_0 x_0$. Let z and z' be the closest points to y and y' on $[x_0, m x_0]$ and $[x_0, m' x_0]$ respectively. Let w and w' be the closest points to $m_0 x_0$ on $[x_0, m x_0]$ and $[x_0, m' x_0]$ respectively. Let x and x' be the points on $[x_0, m x_0]$ and $[x_0, m' x_0]$ respectively at the same distance $a + b - 3\epsilon$ from x_0. Let us prove that x and x' are well defined and satisfy the requirements of the last assertion of Proposition 4.6.





We have

$$\max\{d(m_0x_0, w), d(m_0x_0, w'), d(y, z), d(y', z')\} \leq \epsilon$$

since $[x_0, m_0x_0] \cup [m_0x_0, mx_0]$ is contained in the ϵ-neighbourhood of $[x_0, mx_0]$ and similarly upon replacing m by m'.

By the last preliminary remark, we have

$$b = \min\{d(x_0, \alpha_1 m_1 x_0),\ d(x_0, \alpha'_1 m'_1 x_0)\} \geq N - 2\epsilon \geq 3\epsilon\ .$$

Hence $a + b - 3\epsilon \geq 0$, and the points x_0, w, z and x_0, w', z' are in this order on $[x_0, mx_0]$ and $[x_0, m'x_0]$ respectively. By the triangle inequality, and since closest point maps do not increase the distances, we have

$$\begin{aligned} d(x_0, x) &= a + b - 3\epsilon = d(x_0, m_0x_0) + d(m_0x_0, y) - 3\epsilon \\ &\leq d(x_0, w) + d(w, z) + 2d(m_0x_0, w) + d(y, z) - 3\epsilon \\ &\leq d(x_0, z) = d(x_0, w) + d(w, z) \leq a + b\ . \end{aligned}$$

In particular, $a + b - 3\epsilon \leq d(x_0, z) \leq d(x_0, mx_0)$ and x is well defined. Similarly, x' is well defined.

By the triangle inequality,

$$\begin{aligned} d(x, y) &\leq d(x, z) + d(z, y) = (d(x_0, z) - d(x_0, x)) + d(z, x) \\ &\leq a + b - (a + b - 3\epsilon) + \epsilon = 4\epsilon\ . \end{aligned}$$

Similarly, $d(x', y') \leq 4\epsilon$, hence again by the triangle inequality

$$d(x, x') \geq d(y, y') - d(x, y) - d(x', y') \geq d(y, y') - 8\epsilon\ .$$

Let us prove that $d(y, y') \geq 1$, which implies the last assertion of Proposition 4.6.

Assume for a contradiction that $d(y, y') < 1$. Let p and p' be the closest points to $m_0\alpha_1x_0$ and $m_0\alpha'_1x_0$ on $[m_0x_0, mx_0]$ and $[m_0x_0, m'x_0]$ respectively. We have

$$\max\{d(m_0\alpha_1x_0, p),\ d(m_0\alpha'_1x_0, q')\} \leq \epsilon\ .$$

Up to permuting m and m', since $y = mx_0$ or $y' = m'x_0$ by the definition of y and y', we may assume that $p' \in [m_0x_0, y']$. Let q be the closest point to p' on $[m_0x_0, mx_0]$. By convexity, we have $d(p', q) \leq d(y', y)$. By the triangle inequality, since closest point maps do not increase distances, and by the hypothesis (1), we have

$$\begin{aligned} d(p, q) &= |d(m_0x_0, p) - d(m_0x_0, q)| \leq |d(m_0x_0, p) - d(m_0x_0, p')| + d(p', q) \\ &\leq |d(m_0x_0, m_0\alpha_1x_0) - d(m_0x_0, m_0\alpha'_1x_0)| + \epsilon + d(p', q) \\ &< ((N + 1) - N) + \epsilon + d(y', y) = 1 + \epsilon + d(y', y)\ . \end{aligned}$$

Therefore

$$\begin{aligned} d(\alpha_1 x_0, \alpha'_1 x_0) = d(m_0\alpha_1 x_0, m_0\alpha'_1 x_0) &\leq d(m_0\alpha_1 x_0, p) + d(p,q) + d(q,p') + d(p', m_0\alpha'_1 x_0) \\ &\leq 1 + 3\epsilon + 2d(y', y) < 6 \leq C\ . \end{aligned}$$

This contradicts the hypothesis (2). This ends the proof of Proposition 4.6. □

We now construct (in Proposition 4.7) finite subsets S of Γ satisfying the assumptions (1)-(3) of Proposition 4.6. These sets S satisfy another property (4), which will allow us to prove in Theorem 4.8 that the critical exponent $\delta_{G,\,F}$, where G is the subsemigroup generated by S, is close to the critical exponent $\delta_{\Gamma,\,F}$.

We will follow arguments contained in the proof of [OtP, Theo. 1], though the necessity of controlling the integral of the potential $\widetilde{F}$ along pieces of orbits of the geodesic flow, and the new application Theorem 4.8, require a much more precise construction. The origin of the techniques goes back to the paper [BiJ], as extended independently by U. Hamenstädt (unpublished) and [Pau] to prove that the Hausdorff dimension of the radial limit set $\Lambda_c\Gamma$ of Γ is the (standard) critical exponent of Γ.

Proposition 4.7 *Assume that $\delta_{\Gamma,\,F} > 0$. For all $\delta' \in \,]0, \delta_{\Gamma,\,F}[$ and $\theta, C, L > 0$, there exist $x_0 \in \widetilde{M}$, $v_0 \in T^1_{x_0}\widetilde{M}$, $N > L$ and S a finite subset of Γ such that*

(1) $N \leq d(x_0, \alpha x_0) < N + 1$ for every $\alpha \in S$;

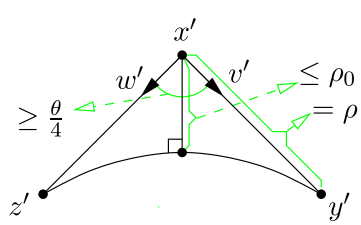
(2) $d(\alpha x_0, \beta x_0) \geq C$ for all distinct α and β in S;

(3) for every $\alpha \in S$, the angle at x_0 between v_0 and the unit tangent vector at x_0 pointing towards αx_0, and the angle at αx_0 between $-\alpha v_0$ and the unit tangent vector at αx_0 pointing towards x_0, are at most θ;

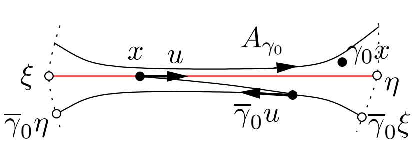
(4) $\displaystyle\sum_{\alpha \in S} e^{\int_{x_0}^{\alpha x_0} \widetilde{F}} \geq e^{\delta' N}$.

Proof. Let us fix positive real constants $\delta' < \delta = \delta_{\Gamma,\,F}$ and θ, C, L. We may assume that $\theta < \frac{\pi}{2}$. Let $\epsilon, \theta' > 0$ be small enough parameters, and let L' be a large enough parameter.

Since $\widetilde{M}$ is CAT(-1), there exist $\rho_0 = \rho_0(\theta) > 0$ and $\rho = \rho(\theta, \theta') \geq \rho_0 + 1$ such that for every $x' \in \widetilde{M}$, for all $v', w' \in T^1_{x'}\widetilde{M}$ whose angle $\angle_{x'}(v', w')$ at x' is a least $\frac{\theta}{4}$, if $y' = \pi(\phi_\rho v')$ and $z' = \pi(\phi_\rho w')$, then the angles at y' and z' of the geodesic triangle with vertices x', y', z' are at most θ', and the distance from x' to the opposite side $[y', z']$ is at most ρ_0.



Since Γ is nonelementary, there exist two distinct points ξ and η in $\Lambda\Gamma$ such that the orbit of (ξ, η) in $\Lambda\Gamma \times \Lambda\Gamma$ under the diagonal action of Γ is dense in $\Lambda\Gamma \times \Lambda\Gamma$. Let u be a unit tangent vector to the geodesic line from ξ to η, and let $x = \pi(u)$.



Again since Γ is nonelementary, the pairs of endpoints of translation axes of hyperbolic elements of Γ are dense in $\Lambda\Gamma \times \Lambda\Gamma$. In particular there exists $\gamma_0 \in \Gamma$ whose translation axis Axe_{γ_0} has endpoints close to ξ and η. Up to replacing γ_0 by a positive or negative power, we may assume that γ_0 translates from the endpoint close to ξ towards the endpoint close to η, and that the geodesic segment $[x, \gamma_0 x]$ has length at least L' and is contained in the ϵ-neighbourhood of Axe_{γ_0}. Denote by u' the unit tangent vector at x pointing towards $\gamma_0 x$ and by u'' the unit tangent vector at $\gamma_0 x$ pointing towards x. We may also assume that the angles $\measuredangle_x(u, u')$ and $\measuredangle_{\gamma_0 x}(\gamma_0 u, -u'')$ are at most $\frac{\theta}{8}$. In particular, by the spherical triangle inequality, we have

$$\angle_{\gamma_0 x}(u'', -\gamma_0 u') \leq \angle_{\gamma_0 x}(u'', -\gamma_0 u) + \angle_{\gamma_0 x}(-\gamma_0 u, -\gamma_0 u') \leq 2\,\frac{\theta}{8} \leq \frac{\theta}{2}\,. \tag{64}$$

By density, there exists $\overline{\gamma}_0 \in \Gamma$ such that $\overline{\gamma}_0 \xi$ is close to η, $\overline{\gamma}_0 \eta$ is close to ξ, the geodesic segment $[x, \overline{\gamma}_0 x]$ has length at least L', and, with $\overline{u}'$ the unit tangent vector at x pointing towards $\overline{\gamma}_0 x$ and $\overline{u}''$ the unit tangent vector at $\overline{\gamma}_0 x$ pointing towards x, the angles $\measuredangle_x(u, \overline{u}')$ and $\measuredangle_{\overline{\gamma}_0 x}(\overline{\gamma}_0 u, \overline{u}'')$ are at most $\frac{\theta}{8}$.

Since $\widetilde{M}$ has pinched negative curvature and ρ depends only on θ, θ', there exists $\theta'' = \theta''(\epsilon, \theta, \theta') \in \,]0, \frac{\theta}{8}]$ such that for all v', v'' in $T^1_x \widetilde{M}$ with angle $\measuredangle_x(v', v'')$ at most θ'', for every $t \in [0, \rho]$ (or equivalently for $t = \rho$ by convexity), we have

$$d(\pi(\phi_t v'), \pi(\phi_t v'')) \leq \epsilon\,.$$

Since Γ is nonelementary, there exists $N_1 \in \mathbb{N} - \{0\}$ such that, with

$$A_n = \{y \in \widetilde{M}\ :\ n \leq d(x, y) < n + N_1\}$$

for every $n \in \mathbb{N}$, the cardinality of $A_n \cap \Gamma x$ tends to $+\infty$ as $n \to +\infty$.

Let us prove that

$$\limsup_{n \to +\infty} \sum_{\gamma \in \Gamma\,:\,\gamma x \in A_n} e^{\int_x^{\gamma x} (\widetilde{F} - \delta')} = +\infty\,.$$

Otherwise there exists $c_1 > 0$ such that, for every $n \in \mathbb{N}$, the above sum is bounded by c_1. If $\delta' < s < \delta$, we have

$$\begin{aligned} Q_{\Gamma,\,F,\,x,\,x}(s) &\leq \sum_{n \in \mathbb{N}}\ \sum_{\gamma \in \Gamma\,:\,\gamma x \in A_n} e^{\int_x^{\gamma x} (\widetilde{F} - s)} = \sum_{n \in \mathbb{N}}\ \sum_{\gamma \in \Gamma\,:\,\gamma x \in A_n} e^{\int_x^{\gamma x} (\widetilde{F} - \delta')} e^{-(s - \delta') d(x, \gamma x)} \\ &\leq c_1 \sum_{n \in \mathbb{N}} e^{-(s - \delta') n} < +\infty\,, \end{aligned}$$

which contradicts the divergence of the Poincaré series $Q_{\Gamma,\,F,\,x,\,x}(s)$ at $s < \delta$.

Since the isometric action of Γ on $\widetilde{M}$ is proper, the group Γ is the union of finitely many subsets E such that

$$d(\alpha x, \beta x) \geq C' = C + \max\{\,2\,\rho_0,\ 2\,d(x, \gamma_0 x),\ 2\,d(x, \overline{\gamma}_0 x)\,\} \tag{65}$$

for all $\alpha \neq \beta$ in E, and the cardinality of $Ex \cap A_n$ tends to $+\infty$ as $n \to +\infty$. There exists such a subset E_0 satisfying

$$\limsup_{n \to +\infty} \sum_{\gamma \in E_0\,:\,\gamma x \in A_n} e^{\int_x^{\gamma x} (\widetilde{F} - \delta')} = +\infty\,.$$

Consider the sequence of the finite subsets Z_n of the unit tangent vectors at x pointing towards the elements of $E_0 x \cap A_n$, for $n \in \mathbb{N}$. By looking at the accumulation points of these subsets and using the compactness of $T^1_x \widetilde{M}$, there exist v, w in $T^1_x \widetilde{M}$, an increasing sequence $(n_k)_{k\in\mathbb{N}}$ in $[2\,\rho_0, +\infty[\, \cap \mathbb{N}$ and, for every $k \in \mathbb{N}$, a finite subset S_k of Γ such that

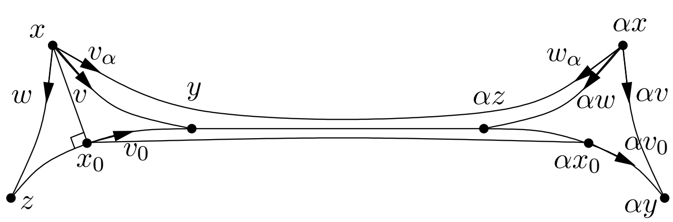
(i) for all α and β in S_k, we have $n_k \leq d(x, \alpha x) \leq n_k + N_1$ and $d(\alpha x, \beta x) \geq C'$ if $\alpha \neq \beta$;

(ii) for every $\alpha \in S_k$, the angle at x between v and the unit tangent vector v_α at x pointing towards αx, and the angle at αx between αw and the unit tangent vector w_α at αx pointing towards x, are at most θ'';

(iii) $\displaystyle\sum_{\alpha\in S_k} e^{\int_x^{\alpha x} \widetilde{F}} \geq k\, e^{\delta' n_k}$.

The end of the proof of Proposition 4.7, which would be easy if $v = -w$, is subdivided in a few cases, the first of them being the easier case when $v = -w$ almost holds, and the other ones dealing with the situation when the equality $v = -w$ is far from holding, and where we try to go back to the easier case by rounding up finitely many geodesic segments in M with the same origin and endpoint, and making them composable in an almost geodesic way.

Case 1. First assume that $\measuredangle_x(v, w) \geq \frac{\theta}{4}$.



Let $y = \pi(\phi_\rho v)$ and $z = \pi(\phi_\rho w)$. Define x_0 as the closest point to x on $[y, z]$ and v_0 as the unit tangent vector at x_0 pointing towards y. In particular, by the definition of ρ_0, we have $d(x, x_0) \leq \rho_0$.

Let $k \in \mathbb{N}$ and $\alpha, \beta \in S_k$. By the triangle inequality and Assertion (i) above, we have

$$0 < n_k - 2\rho_0 \leq d(x_0, \alpha x_0) \leq n_k + N_1 + 2\rho_0\ , \tag{66}$$

and, if $\alpha \neq \beta$, by the definition of C' in Equation (65),

$$d(\alpha x_0, \beta x_0) \geq C' - 2\rho_0 \geq C\ .$$

If k is large enough, then $d(x, \alpha x$ is large enough by Assertion (i) and, since $\measuredangle_x(v, v_\alpha) \leq \theta''$ and $\measuredangle_{\alpha x}(\alpha w, w_\alpha) \leq \theta''$ by Assertion (ii) above, the points y and αz are, by the definition of θ'', at distance at most ϵ to the geodesic segment $[x, \alpha x]$.

If two geodesic segments in a non-positively curved manifold have lengths at least 1 and have sufficiently close endpoints, then their tangent vectors (parametrising them proportionally to arclength by $[0, 1]$) are uniformly close. In particular (or by using Lemma 3.2), since $\rho \geq 1$, if ϵ is small enough and if k is large enough, there exists a constant $c_2 > 0$ (depending only on ϵ, the Hölder constants of $\widetilde{F}$ and $\max_{\pi^{-1}(B(y,\epsilon)\cup B(z,\epsilon))} |\widetilde{F}|$) such that

$$\Big| \int_x^{\alpha x} \widetilde{F} - \int_x^{y} \widetilde{F} - \int_y^{\alpha z} \widetilde{F} - \int_{\alpha z}^{\alpha x} \widetilde{F} \Big| \leq c_2\ . \tag{67}$$

By the triangle inequality and the definition of ρ, we have $d(x_0, y) \geq \rho - \rho_0 \geq 1$, and similarly $d(\alpha z, \alpha x_0) \geq 1$. If k is large enough, we have, for every $\alpha \in S_k$,

$$d(y, \alpha z) \geq d(x, \alpha x) - 2\rho \geq n_k - 2\rho \geq 1 \ .$$

Recall that $d(y, [x, \alpha x]) \leq \epsilon$, $d(\alpha z, [x, \alpha x]) \leq \epsilon$, and that $\measuredangle_y(\phi_\rho v, \phi_{d(x_0,y)} v_0) \leq \theta'$ by the definition of ρ. Let $\epsilon' > 0$ be small enough, and assume that θ' and ϵ are small enough compared to θ and ϵ'. Then, if k is large enough, the piecewise geodesic path

$$\omega = [x_0, y] \cup [y, \alpha z] \cup [\alpha z, \alpha x_0]$$

has its exteriors angles at y and αz at most $\theta_0(1, \epsilon', \theta)$, where $\theta_0(\cdot, \cdot, \cdot)$ has been defined in Lemma 2.6. Therefore by this lemma, for every $\alpha \in S_k$, the angles at x_0 between v_0 and the unit tangent vector at x_0 pointing towards αx_0, and the angle at αx_0 between $-\alpha v_0$ and the unit tangent vector at αx_0 pointing towards x_0, are at most θ; furthermore the piecewise geodesic path ω is at distance at most ϵ' from $[x_0, \alpha x_0]$.

As in the proof of Equation (67), there exists a constant $c_3 > 0$ (depending only on ϵ', the Hölder constants of $\widetilde{F}$ and $\max_{\pi^{-1}(B(y,\,\epsilon')\,\cup\, B(z,\,\epsilon'))} |\widetilde{F}|$) such that

$$\Big| \int_{x_0}^{\alpha x_0} \widetilde{F} - \int_{x_0}^{y} \widetilde{F} - \int_{y}^{\alpha z} \widetilde{F} - \int_{\alpha z}^{\alpha x_0} \widetilde{F} \,\Big| \leq c_3 \ . \tag{68}$$

With $c_4 = \rho \max_{\pi^{-1}(B(x,\,\rho))} |\widetilde{F}|$, since $d(x_0, y) \leq d(x, y) = \rho$ and similarly with z instead of y, we have

$$\Big| \int_{x_0}^{y} \widetilde{F} \,\Big|, \ \ \Big| \int_{x}^{y} \widetilde{F} \,\Big|, \ \ \Big| \int_{\alpha z}^{\alpha x} \widetilde{F} \,\Big|, \ \ \Big| \int_{\alpha z}^{\alpha x_0} \widetilde{F} \,\Big| \leq c_4 \ . \tag{69}$$

By the formulas (67), (68), (69), and by Assertion (iii) above, we have, if k is large enough,

$$\sum_{\alpha \in S_k} e^{\int_{x_0}^{\alpha x_0} \widetilde{F}} \ \geq e^{-c_2 - c_3 - 4c_4} \sum_{\alpha \in S_k} e^{\int_{x}^{\alpha x} \widetilde{F}} \ \geq k \ e^{-c_2 - c_3 - 4c_4} \ e^{\delta' n_k} \ .$$

Hence, using Equation (66), there exists $N \in \mathbb{N} \cap \,]n_k - 2\rho_0 - 1, n_k + N_1 + 2\rho_0 + 1[$ such that

$$\sum_{\substack{\alpha \,\in\, S_k \\ N \,\leq\, d(x_0,\, \alpha x_0) \,<\, N+1}} e^{\int_{x_0}^{\alpha x_0} \widetilde{F}} \ \geq \frac{k \ e^{-c_2 - c_3 - 4c_4}}{N_1 + 4\rho_0 + 2} \ e^{\delta' n_k} \geq \frac{k \ e^{-c_2 - c_3 - 4c_4 - \delta'(N_1 + 2\rho_0 + 1)}}{N_1 + 4\rho_0 + 2} \ e^{\delta' N} \ .$$

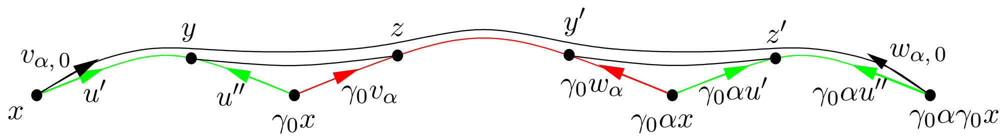
If k is large enough, then $\frac{k \ e^{-c_2 - c_3 - 4c_4 - \delta'(N_1 + 2\rho_0 + 1)}}{N_1 + 4\rho_0 + 2} \geq 1$ and $N \geq n_k - 2\rho_0 - 1 > L$, and defining

$$S = \{\alpha \in S_k \ : \ N \leq d(x_0, \alpha x_0) < N + 1\} \ ,$$

the objects x_0, v_0, N, S defined above satisfy the properties (1)-(4) required in Proposition 4.7.

Case 2. Assume that $\measuredangle_x(u, -v) \geq \frac{\theta}{2}$ and that $\measuredangle_x(u, w) \geq \frac{\theta}{2}$.

Consider the hyperbolic element γ_0 of Γ and the unit tangent vectors u' and u'' defined above. We will define in due time the notation in the following figure.



Let $k \in \mathbb{N}$ and $\alpha \in S_k$. In order to get the following inequalities, we use:

- the spherical triangle inequality for the first one;
- the assumption of Case 2, the properties of u' and u'', and Assertion (ii) above for the second one;
- and the assumption on θ'' for the last one:

$$\measuredangle_{\gamma_0 x}(u'', \gamma_0 v_\alpha) \geq \measuredangle_{\gamma_0 x}(-\gamma_0 u, \gamma_0 v) - \measuredangle_{\gamma_0 x}(-\gamma_0 u, u'') - \measuredangle_{\gamma_0 x}(\gamma_0 v_\alpha, \gamma_0 v)$$
$$\geq \frac{\theta}{2} - \frac{\theta}{8} - \theta'' \geq \frac{\theta}{4}\,.$$

Similarly, $\measuredangle_{\gamma_0 \alpha x}(\gamma_0 w_\alpha, \gamma_0 \alpha u') \geq \frac{\theta}{4}$.

Hence, by the definition of ρ, the angles at $y = \pi(\phi_\rho u'')$ and $z = \pi(\phi_\rho(\gamma_0 v_\alpha))$ of the geodesic triangle with vertices $\gamma_0 x, y, z$ are at most θ', and similarly the angles at $y' = \pi(\phi_\rho(\gamma_0 w_\alpha))$ and $z' = \pi(\phi_\rho(\gamma_0 \alpha u'))$ of the geodesic triangle with vertices $\gamma_0 \alpha x, y', z'$ are at most θ'.

Assume that $L' \geq \rho + 1$, so that by the construction of γ_0

$$d(x, y) = d(x, \gamma_0 x) - \rho \geq L' - \rho \geq 1\,,$$

and similarly $d(z', \gamma_0 \alpha \gamma_0 x) \geq 1$. By the triangle inequality and the definition of ρ, we have $d(y, z) \geq 2(\rho - \rho_0) \geq 2$, and similarly $d(y', z') \geq 2$. Assume that k is large enough, in particular so that we have

$$d(z, y') = d(\gamma_0 x, \gamma_0 \alpha x) - d(\gamma_0 x, z) - d(y', \gamma_0 \alpha x) \geq n_k - 2\rho \geq 1\,.$$

Let $\epsilon' > 0$ be small enough, and assume that θ' is small enough, so that in particular $\theta' \leq \theta_0(1, \epsilon', \frac{\theta}{2})$. The piecewise geodesic path

$$\omega = [x, y] \cup [y, z] \cup [z, y'] \cup [y', z'] \cup [z', \gamma_0 \alpha \gamma_0 x]$$

has exterior angles at y, z, y', z' at most θ'. Hence, by Lemma 2.6, the angles at x between u' and the unit tangent vector $v_{\alpha,0}$ at x pointing towards $\gamma_0 \alpha \gamma_0 x$, and the angle at $\gamma_0 \alpha \gamma_0 x$ between $\gamma_0 \alpha u''$ and the unit tangent vector $w_{\alpha,0}$ at $\gamma_0 \alpha \gamma_0 x$ pointing towards x, are at most $\frac{\theta}{2}$:

$$\measuredangle_x(u', v_{\alpha,0}) \leq \frac{\theta}{2}, \quad \measuredangle_{\gamma_0 \alpha \gamma_0 x}(\gamma_0 \alpha u'', w_{\alpha,0}) \leq \frac{\theta}{2}\,. \tag{70}$$

Furthermore the piecewise geodesic path ω is at distance at most ϵ' from $[x, \gamma_0 \alpha \gamma_0 x]$.

Hence, as in Case 1, there exists a constant $c_5 > 0$ (depending only on ϵ' and on the Hölder constants of $\widetilde{F}$) such that

$$\Big| \int_x^{\gamma_0 \alpha \gamma_0 x} \widetilde{F} - \int_x^y \widetilde{F} - \int_y^z \widetilde{F} - \int_z^{y'} \widetilde{F} - \int_{y'}^{z'} \widetilde{F} - \int_{z'}^{\gamma_0 \alpha \gamma_0 x} \widetilde{F} \Big| \leq c_5\,. \tag{71}$$

By invariance and additivity, we have

$$\int_x^{\alpha x} \widetilde{F} = \int_{\gamma_0 x}^{\gamma_0 \alpha x} \widetilde{F} = \int_{\gamma_0 x}^z \widetilde{F} + \int_z^{y'} \widetilde{F} + \int_{y'}^{\gamma_0 \alpha x} \widetilde{F}\,. \tag{72}$$

Note that $d(x,y) \leq d(x,\gamma_0 x)$, $d(y,z) \leq 2\rho$ and $d(y',z') \leq 2\rho$. Therefore, with $c_6 = (d(x,\gamma_0 x)+2\rho) \max_{\pi^{-1}(B(x,\,d(x,\gamma_0 x)+2\rho))} |\widetilde{F}|$, we have

$$\Big|\int_{\gamma_0 x}^{z} \widetilde{F}\,\Big|,\ \Big|\int_{y'}^{\gamma_0\alpha x} \widetilde{F}\,\Big|,\ \Big|\int_{x}^{y} \widetilde{F}\,\Big|,\ \Big|\int_{y}^{z} \widetilde{F}\,\Big|,\ \Big|\int_{y'}^{z'} \widetilde{F}\,\Big|,\ \Big|\int_{z'}^{\gamma_0\alpha\gamma_0 x} \widetilde{F}\,\Big| \leq c_6\ .$$

Hence, by the formulas (71) and (72), we have

$$\int_{x}^{\gamma_0\alpha\gamma_0 x} \widetilde{F} \geq \Big(\int_{x}^{\alpha x} \widetilde{F}\,\Big) - c_5 - 6c_6\ .$$

Assertion (iii) above implies that

$$\sum_{\alpha\in S_k} e^{\int_x^{\gamma_0\alpha\gamma_0 x}\widetilde{F}} \geq k\; e^{-c_5-6c_6}\; e^{\delta' n_k}\ .$$

Define

$$x_0 = x \quad\text{and}\quad v_0 = u'\ .$$

By Equation (70), for k large enough and for every $\alpha \in S_k$, we have

$$\measuredangle_{x_0}(v_0, v_{\alpha,0}) \leq \frac{\theta}{2} \leq \theta\ .$$

By the spherical triangle inequality and since the isometry $\gamma_0\alpha$ preserves the angles for the first inequality, and by the formulas (64) and (70) for the second one, we have

$$\measuredangle_{\gamma_0\alpha\gamma_0 x_0}(-\gamma_0\alpha\gamma_0 u', w_{\alpha,0}) \leq \measuredangle_{\gamma_0 x}(-\gamma_0 u', u'') + \measuredangle_{\gamma_0\alpha\gamma_0 x}(\gamma_0\alpha u'', w_{\alpha,0}) \leq \frac{\theta}{2}+\frac{\theta}{2} = \theta\ .$$

For k large enough and for all $\alpha,\beta \in S_k$, we have by the triangle inequality and Assertion (i) above

$$n_k - 2\,d(x,\gamma_0 x) \leq d(x_0, \gamma_0\alpha\gamma_0 x_0) \leq n_k + N_1 + 2\,d(x,\gamma_0 x)\ ,$$

and, if $\alpha \neq \beta$, by the definition of C' in Equation (65),

$$\begin{aligned} d(\gamma_0\alpha\gamma_0 x_0, \gamma_0\beta\gamma_0 x_0) &= d(\alpha\gamma_0 x, \beta\gamma_0 x) \geq d(\alpha x, \beta x) - d(\alpha x, \alpha\gamma_0 x) - d(\beta x, \beta\gamma_0 x) \\ &\geq C' - 2\,d(x,\gamma_0 x) \geq C\ . \end{aligned}$$

As in Case 1, there exists $N \in \mathbb{N} \cap\,]n_k - 2\,d(x,\gamma_0 x) - 1, n_k + N_1 + 2\,d(x,\gamma_0 x)+1[$ such that

$$\begin{aligned} \sum_{\substack{\alpha\,\in\, S_k \\ N \,\leq\, d(x_0,\,\gamma_0\alpha\gamma_0 x_0) \,<\, N+1}} e^{\int_{x_0}^{\gamma_0\alpha\gamma_0 x_0}\widetilde{F}} &\geq \frac{k\; e^{-c_5-6c_6}}{N_1 + 4\,d(x,\gamma_0 x)+2}\; e^{\delta' n_k} \\ &\geq \frac{k\; e^{-c_5-6c_6-\delta'(N_1+2\,d(x,\gamma_0 x)+1)}}{N_1+4\,d(x,\gamma_0 x)+2}\; e^{\delta' N}\ . \end{aligned}$$

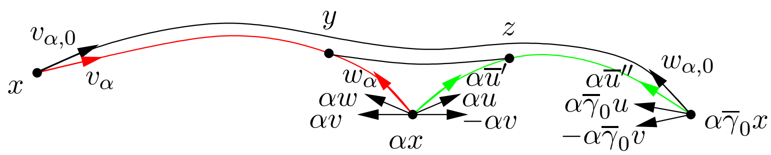
If k is large enough, then $\frac{k\; e^{-c_5-6c_6-\delta'(N_1+2\,d(x,\gamma_0 x)+1)}}{N_1+4\,d(x,\gamma_0 x)+2} \geq 1$ and $N \geq n_k - 2\,d(x,\gamma_0 x) - 1 > L$, and defining

$$S = \{\gamma_0\alpha\gamma_0 \;:\; \alpha \in S_k \ \text{ and }\ N \leq d(x_0, \gamma_0\alpha\gamma_0 x_0) < N+1\}\ ,$$

the objects x_0, v_0, N, S defined above satisfy the properties (1)-(4) required in Proposition 4.7.

Case 3. Assume that $\measuredangle_x(v,w) < \frac{\theta}{4}$ and that $\measuredangle_x(u,-v) < \frac{\theta}{2}$.

Consider the element $\overline{\gamma}_0$ of Γ and the unit tangent vectors $\overline{u}'$ and $\overline{u}''$ defined above.



Let $k \in \mathbb{N}$ and $\alpha \in S_k$. By the spherical triangle inequality, by the assumptions of Case 3, by Assertion (ii) above, by the properties of $\overline{u}'$, and since $\theta'' \leq \frac{\theta}{8}$ and $\theta \leq \frac{\pi}{2}$, we have

$$\begin{aligned}\measuredangle_{\alpha x}(w_\alpha, \alpha\overline{u}') \geq\ & \measuredangle_{\alpha x}(\alpha v, -\alpha v) \\ & - \measuredangle_{\alpha x}(\alpha v, \alpha w) - \measuredangle_{\alpha x}(\alpha w, w_\alpha) - \measuredangle_{\alpha x}(-\alpha v, \alpha u) - \measuredangle_{\alpha x}(\alpha u, \alpha\overline{u}') \\ \geq\ & \pi - \frac{\theta}{4} - \theta'' - \frac{\theta}{2} - \frac{\theta}{8} \geq \pi - \theta \geq \frac{\pi}{2} \geq \frac{\theta}{4}\ .\end{aligned}$$

As in Case 2, if L' and k are large enough, for every ϵ' small enough, if θ' is small enough, so that in particular $L' \geq \rho + 1$, $n_k \geq \rho + 1$, $\theta' \leq \theta_0(1, \epsilon', \frac{\theta}{4})$, with $y = \pi(\phi_\rho w_\alpha)$ and $z = \pi(\phi_\rho(\alpha\overline{u}'))$, considering the piecewise geodesic path

$$w = [x, y] \cup [y, z] \cup [z, \alpha\overline{\gamma}_0 x]\ ,$$

we have for every $\alpha \in S_k$:

- the angles at y and z of the geodesic triangle with vertices $\alpha x, y, z$ are at most θ', by the definition of ρ, hence the exterior angles of ω at y, z are at most $\theta' \leq \theta_0(1, \epsilon', \frac{\theta}{4})$;
- we have

$$d(x,y) \geq n_k - \rho \geq 1\ ,\ \ d(y,z) \geq 2(\rho - \rho_0) \geq 2\ ,$$
$$d(z, \alpha\overline{\gamma}_0 x) = d(\alpha x, \alpha\overline{\gamma}_0 x) - \rho \geq L' - \rho \geq 1\ .$$

hence we may apply Lemma 2.6 to ω with $\ell = 1$;

- the angles at x between v_α and the unit tangent vector $v_{\alpha,0}$ at x pointing towards $\alpha\overline{\gamma}_0 x$, and the angle at $\alpha\overline{\gamma}_0 x$ between $\alpha\overline{u}''$ and the unit tangent vector $w_{\alpha,0}$ at $\alpha\overline{\gamma}_0 x$ pointing towards x, are at most $\frac{\theta}{4}$;
- the path ω is at distance at most ϵ' from $[x, \alpha\overline{\gamma}_0 x]$.
- there exists a constant $c_7 > 0$ (depending only on ϵ' and on the Hölder constants of $\widetilde{F}$) such that

$$\Big|\int_x^{\alpha\overline{\gamma}_0 x} \widetilde{F} - \int_x^y \widetilde{F} - \int_y^z \widetilde{F} - \int_z^{\alpha\overline{\gamma}_0 x} \widetilde{F}\,\Big| \leq c_7\ .$$

- if $c_8 = (d(x, \overline{\gamma}_0 x) + 2\rho)\max_{\pi^{-1}(B(x,\, d(x,\overline{\gamma}_0 x)+2\rho))} |\widetilde{F}|$, we have

$$\Big|\int_y^{\alpha x} \widetilde{F}\,\Big|,\ \Big|\int_y^z \widetilde{F}\,\Big|,\ \Big|\int_z^{\alpha\overline{\gamma}_0 x} \widetilde{F}\,\Big| \leq c_8\ .$$

Since $\int_x^{\alpha x}\widetilde{F}=\int_x^y\widetilde{F}+\int_y^{\alpha x}\widetilde{F}$, and by the last two points, we have

$$\int_x^{\alpha\overline{\gamma}_0 x}\widetilde{F}\ \geq\Big(\int_x^{\alpha x}\widetilde{F}\ \Big)-c_7-3\,c_8\ .$$

Assertion (iii) above implies that

$$\sum_{\alpha\in S_k} e^{\int_x^{\alpha\overline{\gamma}_0 x}\widetilde{F}}\ \geq k\ e^{-c_7-3\,c_8}\ e^{\delta' n_k}\ .$$

Define

$$x_0=x\quad\text{and}\quad v_0=v\ .$$

By Assertion (ii) above and the third point above, we have

$$\measuredangle_{x_0}(v_0,v_{\alpha,0})\leq\measuredangle_x(v,v_\alpha)+\measuredangle_x(v_\alpha,v_{\alpha,0})\leq\theta''+\frac{\theta}{4}\leq\theta\ .$$

By the assumptions of Case 3, by the properties of $\overline{u}''$, and by the third point above, we have

$$\begin{aligned}\measuredangle_{\alpha\overline{\gamma}_0 x}(-\alpha\overline{\gamma}_0 v_0,w_{\alpha,0})&\leq\measuredangle_{\alpha\overline{\gamma}_0 x}(-\alpha\overline{\gamma}_0 v_0,\alpha\overline{\gamma}_0 u)+\measuredangle_{\alpha\overline{\gamma}_0 x}(\alpha\overline{\gamma}_0 u,\alpha\overline{u}'')+\measuredangle_{\alpha\overline{\gamma}_0 x}(\alpha\overline{u}'',w_{\alpha,0})\\&\leq\frac{\theta}{2}+\frac{\theta}{8}+\frac{\theta}{4}\leq\theta\ .\end{aligned}$$

For all $\alpha,\beta\in S_k$, by the triangle inequality and Assertion (i) above, we have

$$n_k-d(x,\overline{\gamma}_0 x)\leq d(x_0,\alpha\overline{\gamma}_0 x_0)\leq n_k+N_1+d(x,\overline{\gamma}_0 x)\ ,$$

and, if $\alpha\neq\beta$, by the definition of C' in Equation (65),

$$d(\alpha\overline{\gamma}_0 x_0,\beta\overline{\gamma}_0 x_0)\geq d(\alpha x,\beta x)-2\,d(x,\overline{\gamma}_0 x)\geq C'-2\,d(x,\overline{\gamma}_0 x)\geq C\ .$$

As in Case 1, there exists $N\in\mathbb{N}\cap\,]n_k-d(x,\overline{\gamma}_0 x)-1,n_k+N_1+d(x,\overline{\gamma}_0 x)+1[$ such that

$$\sum_{\substack{\alpha\,\in\,S_k\\ N\,\leq\,d(x_0,\,\alpha\overline{\gamma}_0 x_0)\,<\,N+1}} e^{\int_{x_0}^{\alpha\overline{\gamma}_0 x_0}\widetilde{F}}\ \geq\frac{k\ e^{-c_7-3c_8-\delta'(N_1+d(x,\overline{\gamma}_0 x)+1)}}{N_1+2\,d(x,\overline{\gamma}_0 x)+2}\ e^{\delta' N}\ .$$

If k is large enough, then $\frac{k\ e^{-c_7-3c_8-\delta'(N_1+d(x,\overline{\gamma}_0 x)+1)}}{N_1+2\,d(x,\overline{\gamma}_0 x)+2}\geq 1$ and $N\geq n_k-d(x,\overline{\gamma}_0 x)-1>L$, and defining

$$S=\{\alpha\overline{\gamma}_0\ :\ \alpha\in S_k\ \text{ and }\ N\leq d(x_0,\alpha\overline{\gamma}_0 x_0)<N+1\}\ ,$$

the objects x_0,v_0,N,S defined above satisfy the properties (1)-(4) required in Proposition 4.7.

Case 4. The remaining case, when $\measuredangle_x(v,w)<\frac{\theta}{4}$ and $\measuredangle_x(u,w)<\frac{\theta}{2}$, follows similarly to Case 3: we take $x_0=x,v_0=w$ and S the appropriate subset of $\{\overline{\gamma}_0\alpha\ :\ \alpha\in S_k\}$.

This ends the proof of Proposition 4.7. □

The main objective of this subsection is the following result.

Theorem 4.8 *Let $\widetilde{M}$ be a complete simply connected Riemannian manifold, with dimension at least 2 and pinched negative sectional curvature at most -1. Let Γ be a nonelementary discrete group of isometries of $\widetilde{M}$. Let $\widetilde{F} : T^1\widetilde{M} \to \mathbb{R}$ be a Hölder-continuous Γ-invariant map.*

Then the critical exponent $\delta_{\Gamma,\,F}$ is equal to the upper bound of the critical exponents $\delta_{G,\,F}$, where G ranges over the convex-cocompact free subsemigroups of Γ.

When $F = 0$, this result (with $\widetilde{M}$ any proper geodesic CAT(-1)-space) is due to Mercat [Mer].

Proof. It is immediate that $\delta_{\Gamma,\,F} \geq \delta_{G,\,F}$ for any subsemigroup G of Γ. Hence $\delta_{\Gamma,\,F}$ is at least the upper bound of the critical exponents $\delta_{G,\,F}$ of the convex-cocompact free subsemigroups G of Γ. We now prove the converse inequality. Up to adding a constant to F, we may assume that $\delta_{\Gamma,\,F} > 0$.

Let $\delta' \in \,]0, \delta_{\Gamma,\,F}[$ and $s \in \,]0, \delta'[$. Since two geodesic segments of lengths at least 1 which are close enough have their unit tangent vectors close (see Lemma 3.2), there exist $\epsilon \in \,]0, 1]$ and $c > 0$ (depending only on the Hölder constants of $\widetilde{F}$ and on $\max_{\pi^{-1}(B(y,1))} |\widetilde{F} - s|$) such that for every piecewise geodesic segment $[x, y] \cup [y, z]$, with segments of length at least 1, which is contained in the ϵ-neighbourhood of $[x, z]$, we have

$$\Big| \int_x^z (\widetilde{F} - s) \;-\; \int_x^y (\widetilde{F} - s) \;-\; \int_y^z (\widetilde{F} - s) \,\Big| \leq c\;.$$

Let $\theta > 0$ (depending only on ϵ) be given by Proposition 4.6. Let $L = \max\{5, \frac{c+s}{\delta'-s}\}$ and $C = 6$. Let $x_0 \in \widetilde{M}$, $v_0 \in T^1\widetilde{M}$, $N > L$ and S a finite subset of Γ be given by Proposition 4.7 (depending on δ', θ, L, C). By Proposition 4.6, whose assumptions are satisfied by the first three assertions of Proposition 4.7, the semigroup G generated by S is free on S and convex-cocompact, and for all $\gamma \in G$ and $\alpha \in S$, the piecewise geodesic segment $[x_0, \gamma x_0] \cup [\gamma x_0, \gamma\alpha x_0]$, whose segments have length at least 1, is contained in the ϵ-neighbourhood of $[x_0, \gamma\alpha x_0]$. In particular, by the definition of c and by invariance, we have

$$\Big| \int_{x_0}^{\gamma\alpha x_0} (\widetilde{F} - s) \;-\; \int_{x_0}^{\gamma x_0} (\widetilde{F} - s) \;-\; \int_{x_0}^{\alpha x_0} (\widetilde{F} - s) \,\Big| \leq c\;. \tag{73}$$

For every $k \in \mathbb{N}$, denote by S_k the set of words of length k in elements of S. Let

$$\Sigma_k = \sum_{\gamma \in S_k} e^{\int_{x_0}^{\gamma x_0} (\widetilde{F} - s)}\;,$$

so that, since G is the free semigroup generated by S, we have

$$Q_{G,\,F,\,x_0,\,x_0}(s) = \sum_{k \in \mathbb{N}} \Sigma_k\;.$$

By Assertion (1) of Proposition 4.7, for every $\alpha \in S$, we have $-sd(x_0, \alpha x_0) \geq -s(N+1)$. By Equation (73), by Assertion (4) of Proposition 4.7, and since $N > L \geq \frac{c+s}{\delta'-s}$, we hence have

$$\begin{aligned}\Sigma_{k+1} = \sum_{\gamma \in S_k,\, \alpha \in S} e^{\int_{x_0}^{\gamma\alpha x_0} (\widetilde{F} - s)} &\geq e^{-c} \sum_{\gamma \in S_k,\, \alpha \in S} e^{\int_{x_0}^{\gamma x_0} (\widetilde{F} - s) + \int_{x_0}^{\alpha x_0} (\widetilde{F} - s)} \\ &= e^{-c}\Sigma_k \sum_{\alpha \in S} e^{\int_{x_0}^{\alpha x_0} (\widetilde{F} - s)} \geq e^{-c-s}\Sigma_k\, e^{(\delta'-s)N} \geq \Sigma_k\;.\end{aligned}$$

Therefore the Poincaré series $Q_{G,F,x_0,x_0}(s)$ diverges. Hence $\delta_{G,F} \geq s$. Letting $s \to \delta'$, this implies that $\delta_{G,F} \geq \delta'$. As $\delta' < \delta_{\Gamma,F}$ was arbitrary, this proves Theorem 4.8. □

5 A Hopf-Tsuji-Sullivan-Roblin theorem for Gibbs states and applications

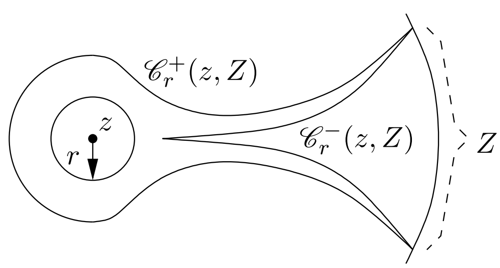
Let $\widetilde{M}, \Gamma, \widetilde{F}$ be as in the beginning of Chapter 2: $\widetilde{M}$ is a complete simply connected Riemannian manifold, with dimension at least 2 and pinched sectional curvature at most -1; Γ is a nonelementary discrete group of isometries of $\widetilde{M}$; and $\widetilde{F} : T^1\widetilde{M} \to \mathbb{R}$ is a Hölder-continuous Γ-invariant map. The aim of this chapter is to give criteria for the ergodicity and non ergodicity of the geodesic flow on the space $T^1M = \Gamma\backslash T^1\widetilde{M}$ endowed with a Gibbs measure. We also discuss several applications, in particular to the uniqueness of a Gibbs measure on T^1M as soon as there exists a finite such measure.

5.1 Some geometric notation

We introduce in this subsection the notation for several geometric objects that will be used in the proofs of the subsections 5.2 and 9.1.

For every $z \in \widetilde{M}$, every subset Z of $\partial_\infty\widetilde{M}$ and every $r > 0$, define the *r-thickened* and *r-thinned* cones over Z with vertex z as

$$\mathscr{C}_r^+(z, Z) = \mathscr{N}_r\Big(\bigcup_{z' \in B(z,r)} \mathscr{C}_{z'}Z\Big) \quad\text{and}\quad \mathscr{C}_r^-(z, Z) = \mathscr{N}_r^-\Big(\bigcap_{z' \in B(z,r)} \mathscr{C}_{z'}Z\Big)\,.$$

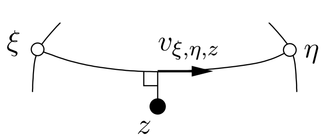


For all $r > 0$ and $z \in \widetilde{M}$, we will say that $v \in T^1\widetilde{M}$ is *r-close to z* if there exists $s \in \,]-\frac{r}{2}, \frac{r}{2}[$ such that $\pi(\phi_s v)$ is the closest point to z on the geodesic line defined by v, and $d(z, \pi(\phi_s v)) < r$. For every $\gamma \in \Gamma$, the vector γv is r-close to γz if and only if v is r-close to z.

For every subset Z of $\partial_\infty\widetilde{M}$, let $K^+(z, r, Z)$ be the set of $v \in T^1\widetilde{M}$ such that $v_+ \in Z$ and v is r-close to z, and $K^-(z, r, Z) = \iota K^+(z, r, Z)$. Note that $\gamma K^\pm(z, r, Z) = K^\pm(\gamma z, r, \gamma Z)$ for every $\gamma \in \Gamma$. Also denote by

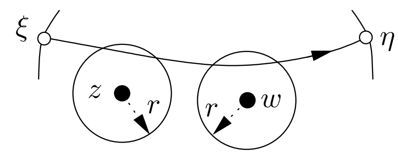
$$K(z, r) = K^+(z, r, \partial_\infty\widetilde{M}) = K^-(z, r, \partial_\infty\widetilde{M})$$

the relatively compact open set of vectors v that are r-close to z, which satisfies $\gamma K(z, r) = K(\gamma z, r)$ for every isometry γ of $\widetilde{M}$.



For distinct points ξ and η in $\partial_\infty\widetilde{M}$, let $v_{\xi,\eta,z}$ be the unit tangent vector based at the closest point to z on the geodesic line betweeen ξ and η pointing towards η. We have $\gamma v_{\xi,\eta,z} = v_{\gamma\xi,\gamma\eta,\gamma z}$ for every isometry γ of $\widetilde{M}$.

For all $r > 0$ and $z, w \in \widetilde{M}$ such that $d(z,w) > 2r$, define $L_r(z,w)$ as the set of (ξ, η) in $\partial_\infty \widetilde{M} \times \partial_\infty \widetilde{M}$ such that the geodesic line starting from ξ and ending at η meets first $B(z,r)$ and then $B(w,r)$. We have $\gamma L_r(z,w) = L_r(\gamma z, \gamma w)$ for every isometry γ of $\widetilde{M}$.



For all $r > 0$ and $z, w \in \widetilde{M}$, define

$$\mathscr{O}_r^+(z,w) = \bigcup_{z' \in B(z,r)} \mathscr{O}_{z'} B(w,r) \quad \text{and} \quad \mathscr{O}_r^-(z,w) = \bigcap_{z' \in B(z,r)} \mathscr{O}_{z'} B(w,r)\ .$$

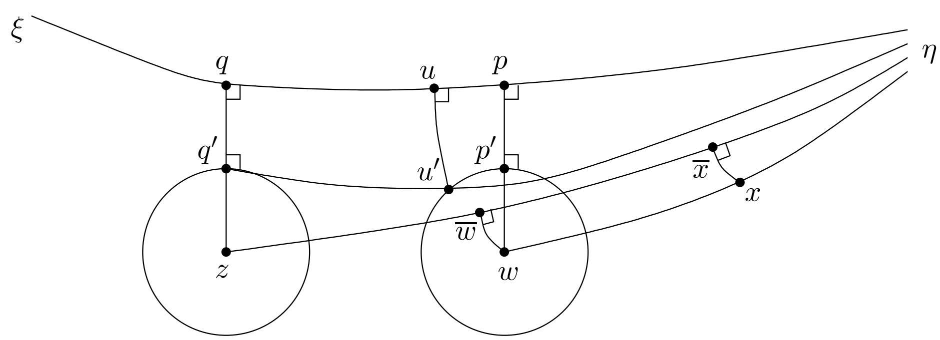
For all $r > 0$, $w \in \widetilde{M}$ and $z \in \partial_\infty \widetilde{M}$, define

$$\mathscr{O}_r^+(z,w) = \mathscr{O}_r^-(z,w) = \mathscr{O}_z B(w,r)\ .$$

Note that, for every fixed $w \in \widetilde{M}$, the closure of $\mathscr{O}_r^\pm(z,w)$ converges to the closure of $\mathscr{O}_r^\pm(z_0,w)$ in the space of closed subsets of $\widetilde{M} \cup \partial_\infty \widetilde{M}$, as z tends to z_0 in $\widetilde{M} \cup \partial_\infty \widetilde{M}$, uniformly on r in a bounded interval. We have $\gamma \mathscr{O}_r^\pm(z,w) = \mathscr{O}_r^\pm(\gamma z, \gamma w)$ for every isometry γ of $\widetilde{M}$.

Lemma 5.1 *For all $r > 0$ and $z, w \in \widetilde{M}$ such that $d(z,w) > 2r$, the set $L_r(z,w)$ contains $\mathscr{O}_r^-(w,z) \times \mathscr{O}_r^-(z,w)$.*

Proof. Let $(\xi, \eta) \in \mathscr{O}_r^-(w,z) \times \mathscr{O}_r^-(z,w)$.



For every x on the geodesic ray $[w, \eta[$, denote by $\overline{x}$ its closest point on the geodesic ray $[z, \eta[$. Since $\eta \in \mathscr{O}_r^-(z,w) \subset \mathscr{O}_z B(w,r)$, the point $\overline{w}$ is at distance at most r from w. Since closest point maps do not increase the distances, by convexity, by the triangle inequality and since $d(z,w) > 2r$, we have

$$d(z,x) \geq d(z,\overline{x}) \geq d(z,\overline{w}) \geq d(z,w) - d(w,\overline{w}) > 2r - r = r\ .$$

Hence $\eta \notin \mathscr{O}_w B(z,r) \supset \mathscr{O}_r^-(w,z)$. This implies that $\xi \neq \eta$.

Let p and q be the closest points to w and z respectively on the geodesic line between ξ and η. Let p' and q' be the points on $[w,p]$ and $[z,q]$ at distance $\min\{r, d(w,p)\}$ and $\min\{r, d(z,q)\}$ from w and z, respectively (see the above picture). Note that p' and q' are the closest points to p and q on the balls $B(w,r)$ and $B(z,r)$ respectively.

Up to a permutation, we may assume that $d(p,w) \geq d(q,z)$. Since $\eta \in \mathscr{O}_r^-(z,w)$, the geodesic ray $[q', \eta[$ meets $B(w,r)$ at at least one point u'. Let u be the closest point to

u' on the geodesic line between ξ and η. Since $d(z,w) > 2r$, we have $u' \neq q'$. Hence, by convexity,

$$d(q,q') \geq d(u,u') \geq d(p,p')\ .$$

If $q \neq q'$, the first inequality is strict, and this contradicts the fact that $d(p,w) = r + d(p,p') \geq d(q,z) = r + d(q,q')$. Hence $q = q'$, therefore $p = p'$ and the points z, w are at distance at most r from the geodesic line $]\xi,\eta[$.

Since $d(q,z) \leq r$ and $\eta \in \mathscr{O}_r^-(z,w)$, the geodesic ray $[q,\eta[$ meets $B(w,r)$. Since $d(z,w) > 2r$, the point q does not belong to the intersection of $B(w,r)$ and $[q,\eta[$. Hence by convexity, p belongs to this intersection, and ξ, q, p, η are in this order on the geodesic line $]\xi,\eta[$. Therefore $(\xi,\eta) \in L_r(z,w)$, as required. □

The following result that we state without proof is a simple extension of Mohsen's shadow lemma 3.10, with a similar proof (see [Rob1, p. 10]).

Lemma 5.2 *Let $\sigma \in \mathbb{R}$, let $(\mu_x)_{x\in\widetilde{M}}$ be a Patterson density of dimension σ for (Γ, F), and let $x, y \in \widetilde{M}$. If R is large enough, there exists $C > 0$ such that for every $\gamma \in \Gamma$,*

$$\frac{1}{C}\, e^{\int_x^{\gamma y}(\widetilde{F}-\sigma)} \leq \mu_x\big(\mathscr{O}_x^- B(\gamma y, R)\big) \leq \mu_x\big(\mathscr{O}_x^+ B(\gamma y, R)\big) \leq C\, e^{\int_x^{\gamma y}(\widetilde{F}-\sigma)}\ .$$

5.2 The Hopf-Tsuji-Sullivan-Roblin theorem for Gibbs states

Fix $\sigma \in \mathbb{R}$. Let $(\mu^\iota_x)_{x\in\widetilde{M}}$ and $(\mu_x)_{x\in\widetilde{M}}$ be Patterson densities of the same dimension σ for $(\Gamma, F\circ\iota)$ and (Γ, F) respectively, and let $\widetilde{m}$ (respectively m) be their associated Gibbs measure on $T^1\widetilde{M}$ (respectively T^1M), see Subsection 3.7.

In this subsection, we give criteria for the ergodicity and non ergodicity of the geodesic flow $(\phi_t)_{t\in\mathbb{R}}$ on $T^1M = \Gamma\backslash T^1\widetilde{M}$ endowed with Gibbs measures. These criteria generalise the classical Hopf-Tsujii-Sullivan-Roblin results for the Bowen-Margulis measures (that is, when $F = 0$). Their proofs, given at the end of this subsection, follow closely the general lines of those presented in [Rob1, Chap. 1E].

Let us recall some definitions on a dynamical system (Ω, G, ν), where Ω is a Hausdorff locally compact σ-compact topological space, G is a locally compact σ-compact topological group acting continuously on Ω, and ν is a positive regular (hence σ-finite) Borel measure on Ω, which is quasi-invariant under G.

Recall that (Ω, G, ν) is *ergodic* if for any G-invariant measurable subset A of Ω, either A or ${}^cA$ has zero measure for ν.

Recall that $\mathbb{1}_A$ denotes the characteristic function of a subset A. A *wandering set* of (Ω, G, ν) is a Borel subset E of Ω such that for ν-almost every $\omega \in E$, the integral $\int_G \mathbb{1}_E(g\omega)\, dg$ is finite, where dg is a fixed left Haar measure on G. The *dissipative part* of (Ω, G, ν) (well defined up to a zero measure subset) is any G-invariant Borel subset which contains every wandering set up to a zero measure subset, and does not contain any nonwandering set of positive measure. Note that any countable union of wandering sets is contained in the dissipative part up to a zero measure subset. The *conservative part* of (Ω, G, ν) is the complement of its dissipative part (and is also defined up to a zero measure subset). Recall that (Ω, G, ν) is *completely conservative* (respectively *completely dissipative*) if its dissipative part (respectively conservative part) has zero measure.

The first result gives criteria for the non-ergodicity of the geodesic flow.

Theorem 5.3 *The following assertions are equivalent.*

(1) The Poincaré series of (Γ, F) *at the point* σ *converges:* $Q_{\Gamma,\,F}(\sigma) < +\infty$.

(2) There exists $x \in \widetilde{M}$ *such that* $\mu_x(\Lambda_c\Gamma) = 0$.

(3) There exists $x \in \widetilde{M}$ *such that* $\mu^\iota_x(\Lambda_c\Gamma) = 0$.

(4) There exists $x \in \widetilde{M}$ *such that the dynamical system* $(\partial^2_\infty\widetilde{M}, \Gamma, (\mu^\iota_x \otimes \mu_x)_{|\partial^2_\infty\widetilde{M}})$ *is non-ergodic and completely dissipative.*

(5) The dynamical system $(T^1M, (\phi_t)_{t\in\mathbb{R}}, m)$ *is non-ergodic and completely dissipative.*

We may replace Assertion (2) by

(2') For every $x \in \widetilde{M}$*, we have* $\mu_x(\Lambda_c\Gamma) = 0$.

and Assertion (3) by

(3') For every $x \in \widetilde{M}$*, we have* $\mu^\iota_x(\Lambda_c\Gamma) = 0$.

since two elements in a Patterson density are pairwise absolutely continuous. The proof will show that we may also replace "There exists $x \in \widetilde{M}$ such that" by "For every $x \in \widetilde{M}$," in Assertion (4).

Remark. This result indicates that the case when $Q_{\Gamma,\,F}(\sigma) < +\infty$ (and in particular the case when (Γ, F) is not of divergence type and $\sigma = \delta_{\Gamma,\,F}$) is not interesting from the dynamical system viewpoint. It is also not the one that occurs the most frequently, as we will see later on. Furthermore, when $Q_{\Gamma,\,F}(\sigma) < +\infty$, uninteresting Patterson densities of dimension σ for (Γ, F) exist if $\Lambda\Gamma \neq \partial_\infty\widetilde{M}$, as follows. Fix x_0 in $\widetilde{M}$ and $\xi \in \partial_\infty\widetilde{M} - \Lambda\Gamma$. Denote by $\mathscr{D}_\eta$ the (unit) Dirac mass at a point $\eta \in \partial_\infty\widetilde{M}$. Then for every $x \in X$, the series

$$\nu^\xi_x = \sum_{\gamma\in\Gamma} e^{-C_{F-\sigma,\,\gamma\xi}(x,\gamma x_0)}\;\mathscr{D}_{\gamma\xi}$$

converges. Indeed, its convergence is independent of x and x_0 by the cocycle property of $C_{F-\sigma}$ (Equation (21)) and by Lemma 3.4 (1). Hence, for every fixed $r > 0$, we may assume that $x \in \mathscr{C}\Lambda\Gamma$ and that x_0 is close enough to ξ so that $\xi \in \mathscr{O}_{\gamma^{-1}x}B(x_0, r)$ for every $\gamma \in \Gamma$. Then by the invariance property of $C_{F-\sigma}$ (Equation (22)) and by Lemma 3.4 (2), there exists $c > 0$ such that for every $\gamma \in \Gamma$, we have

$$-C_{F-\sigma,\,\gamma\xi}(x, \gamma x_0) = -C_{F-\sigma,\,\xi}(\gamma^{-1}x, x_0) \leq c + \int_{\gamma^{-1}x}^{x_0} (\widetilde{F} - \sigma) = c + \int_x^{\gamma x_0} (\widetilde{F} - \sigma)\,.$$

Hence the convergence (in norm) of the series ν^ξ_x follows from the convergence of the series $Q_{\Gamma,\,F,\,x,\,x_0}(\sigma)$. It is easy to check that $(\nu^\xi_x)_{x\in\widetilde{M}}$ is a Patterson density of dimension σ for (Γ, F). Taking linear combinations of them as ξ varies and weak star limits gives other examples. The measures ν^ξ_x are atomic and ergodic (but not properly ergodic, indeed of type I_∞ in the Hopf-Krieger classification, see for instance [FM]: they give full measure to an infinite orbit of Γ).

The next result gives criteria for the ergodicity of the geodesic flow.

Theorem 5.4 *The following assertions are equivalent.*

(i) The Poincaré series of (Γ, F) at the point σ diverges: $Q_{\Gamma,F}(\sigma) = +\infty$.

(ii) There exists $x \in \widetilde{M}$ such that $\mu_x({}^c\Lambda_c\Gamma) = 0$.

(iii) There exists $x \in \widetilde{M}$ such that $\mu^\iota_x({}^c\Lambda_c\Gamma) = 0$.

(iv) There exists $x \in \widetilde{M}$ such that the dynamical system $(\partial^2_\infty\widetilde{M}, \Gamma, (\mu^\iota_x \otimes \mu_x)_{|\partial^2_\infty\widetilde{M}})$ is ergodic and completely conservative.

(v) The dynamical system $(T^1M, (\phi_t)_{t\in\mathbb{R}}, m)$ is ergodic and completely conservative.

We may also replace Assertion (ii) by

(ii') For every $x \in \widetilde{M}$, we have $\mu_x({}^c\Lambda_c\Gamma) = 0$.

and Assertion (iii) by

(iii') For every $x \in \widetilde{M}$, we have $\mu^\iota_x({}^c\Lambda_c\Gamma) = 0$.

The proof will show that we may also replace "There exists $x \in \widetilde{M}$ such that" by "For every $x \in \widetilde{M}$," in Assertion (iv).

The proof of these two theorems will be based on the following list of implications, as in the case $F = 0$ in [Rob1, Chap. 1E]. Let us fix $x \in \widetilde{M}$.

Proposition 5.5 *Assume that $\sigma > 0$.*

(a) If $Q_{\Gamma,F}(\sigma) < +\infty$ then $\mu_x(\Lambda_c\Gamma) = 0$ and $\mu^\iota_x(\Lambda_c\Gamma) = 0$.

(b) If $\mu_x(\Lambda_c\Gamma) = 0$ and $\mu^\iota_x(\Lambda_c\Gamma) = 0$, then $(\partial^2_\infty\widetilde{M}, \Gamma, (\mu^\iota_x \otimes \mu_x)_{|\partial^2_\infty\widetilde{M}})$ is completely dissipative.

(c) If $(\partial^2_\infty\widetilde{M}, \Gamma, (\mu^\iota_x \otimes \mu_x)_{|\partial^2_\infty\widetilde{M}})$ is ergodic, then $\mu^\iota_x \otimes \mu_x$ is non-atomic, the diagonal of $\partial_\infty\widetilde{M} \times \partial_\infty\widetilde{M}$ has zero measure for $\mu^\iota_x \otimes \mu_x$, and $(\partial_\infty\widetilde{M} \times \partial_\infty\widetilde{M}, \Gamma, \mu^\iota_x \otimes \mu_x)$ is ergodic and completely conservative.

(d) For all $v \in T^1\widetilde{M}$, we have $v_+ \in \Lambda_c\Gamma$ if and only if there exists a relatively compact subset K in $T^1\widetilde{M}$ such that $\int_0^\infty \mathbb{1}_{\Gamma K} \circ \phi_t(v)dt$ diverges. Therefore $(T^1M, (\phi_t)_{t\in\mathbb{R}}, m)$ is completely conservative (respectively completely dissipative) if and only if $\mu^\iota_x({}^c\Lambda_c\Gamma) = 0$ and $\mu_x({}^c\Lambda_c\Gamma) = 0$ (respectively $\mu^\iota_x(\Lambda_c\Gamma) = 0$ and $\mu_x(\Lambda_c\Gamma) = 0$).

(e) The dynamical system $(T^1M, (\phi_t)_{t\in\mathbb{R}}, m)$ is ergodic if and only if the dynamical system $(\partial^2_\infty\widetilde{M}, \Gamma, (\mu^\iota_x \otimes \mu_x)_{|\partial^2_\infty\widetilde{M}})$ is ergodic.

(f) If $Q_{\Gamma,F}(\sigma) = +\infty$ then $\mu_x(\Lambda_c\Gamma) > 0$ and $\mu^\iota_x(\Lambda_c\Gamma) > 0$.

(g) If $\mu_x(\Lambda_c\Gamma) > 0$ then $\mu_x({}^c\Lambda_c\Gamma) = 0$.

(h) If $(T^1M, (\phi_t)_{t\in\mathbb{R}}, m)$ is completely conservative then it is ergodic.

Proof. (a) Mohsen's shadow lemma 3.10 applied to $(\mu_x)_{x\in\widetilde{M}}$ with $K = \{x, y\}$ shows the existence of $N_0 \in \mathbb{N}$ such that for all $N \geq N_0$, there exists $c_N > 0$ such that

$$\mu_x(\mathscr{O}_x(B(\gamma y, N))) \leq C_N \; e^{\int_x^{\gamma x}(wtF-\sigma)} \; .$$

By the definition of $\Lambda_c\Gamma$, we have $\Lambda_c\Gamma = \bigcup_{N\in\mathbb{N},\, N\geq N_0} \Lambda(N)$ where

$$\Lambda(N) = \bigcap_{A \text{ finite subset of } \Gamma} \;\; \bigcup_{\gamma\in\Gamma-A} \mathscr{O}_x(B(\gamma y, N)) \;.$$

The fact that $\mu_x(\Lambda(N)) = 0$ for every $N \geq N_0$, which implies that $\mu_x(\Lambda_c\Gamma) = 0$, then follows by using the easy part of the Borel-Cantelli lemma.

Since $Q_{\Gamma,\,F}(\sigma) < +\infty$ if and only if $Q_{\Gamma,\,F\circ\iota}(\sigma) < +\infty$ (see Lemma 3.3 (ii)), the last claim follows similarly.

(b) By Fubini's theorem, the set $A = \{(\xi,\eta) \in \partial^2_\infty\widetilde{M} : \xi,\eta \notin \Lambda_c\Gamma\}$ has full measure with respect to $(\mu^\iota_x \otimes \mu_x)_{|\partial^2_\infty\widetilde{M}}$. For every $(\xi,\eta) \in A$, let $\Gamma_{\xi,\,\eta}$ be the finite nonempty set of $\alpha \in \Gamma$ such that the distance between αx and the geodesic line between ξ and η is minimal; it satisfies $\gamma\Gamma_{\xi,\,\eta} = \Gamma_{\gamma\xi,\,\gamma\eta}$ for every $\gamma \in \Gamma$. For any finite subset F of Γ, we denote $S_F = \{(\xi,\eta) \in A : \Gamma_{\xi,\,\eta} = F\}$. We claim that the measurable subset S_F is wandering, since if γS_F meets S_F then $\gamma F = F$ and since the stabiliser (for the action by left translations of Γ on itself) of a finite subset of Γ is finite. As a countable union of wandering sets is contained in the dissipative part up to a zero measure subset, and since $A = \bigcup_F S_F$ has full measure, the result follows.

(c) Assume for a contradiction that $(\xi',\xi) \in \partial_\infty\widetilde{M} \times \partial_\infty\widetilde{M}$ is an atom for $\mu^\iota_x \otimes \mu_x$, that is, ξ' is an atom for μ^ι_x and ξ is an atom for μ_x. This implies that $\gamma\xi$ is an atom for μ_x for every $\gamma \in \Gamma$ by quasi-invariance. Let $\Gamma_{\xi'}$ be the stabiliser of ξ' in Γ, and note that its limit set $\Lambda\Gamma_{\xi'}$ is finite (it has 0, 1 or 2 points). Since Γ is nonelementary, there exists $\gamma_0 \in \Gamma$ such that $\gamma_0\xi \neq \xi'$. By ergodicity, the Γ-orbit $\mathscr{O}$ of $(\xi',\gamma_0\xi)$ in $\partial^2_\infty\widetilde{M}$ has full measure with respect to $(\mu^\iota_x \otimes \mu_x)_{|\partial^2_\infty\widetilde{M}}$. Since Γ acts minimally in $\Lambda\Gamma$, there exists $\gamma_1 \in \Gamma$ such that $\gamma_1\xi$ does not lie in the countable closed set $\Gamma_{\xi'}(\gamma_0\xi) \cup \Lambda\Gamma_{\xi'} \cup \{\xi'\}$, whose complement in $\Lambda\Gamma$ is open and nonempty since $\Lambda\Gamma$ is uncountable. Note that $(\xi',\gamma_1\xi)$ does not belong to $\mathscr{O}$, since otherwise there exists $\gamma \in \Gamma$ such that $\xi' = \gamma\xi'$ and $\gamma_1\xi = \gamma\gamma_0\xi$, contradicting that $\gamma_1\xi$ does not belong to $\Gamma_{\xi'}(\gamma_0\xi)$. But $(\xi',\gamma_1\xi)$ is an atom of $(\mu^\iota_x \otimes \mu_x)_{|\partial^2_\infty\widetilde{M}}$, contradicting the fact that $\mathscr{O}$ has full measure with respect to this measure.

Since the product measure $\mu^\iota_x \otimes \mu_x$ is nonatomic, the diagonal of $\partial_\infty\widetilde{M} \times \partial_\infty\widetilde{M}$ has measure zero for the product measure (by Fubini's theorem, since a finite measure has at most countable many atoms). Hence the dynamical system $(\partial_\infty\widetilde{M} \times \partial_\infty\widetilde{M}, \Gamma, \mu^\iota_x \otimes \mu_x)$ is also ergodic. Since Γ is countable and since $\mu^\iota_x \otimes \mu_x$ is nonatomic, this ergodic dynamical system is completely conservative (see for instance [Aar, page 51, Prop. 1.6.6]).

(d) The first claim that the conical limit points are the endpoints of the geodesic rays whose image in $M = \Gamma\backslash\widetilde{M}$ returns infinitely often in some relatively compact subset is immediate from the definition of $\Lambda_c\Gamma$. It follows that if E is a relatively compact wandering set in T^1M, then for $\widetilde{m}$-almost every $v \in T^1\widetilde{M}$, the image of v in T^1M belongs to E if and only if neither v_- nor v_+ belong to $\Lambda_c\Gamma$. The result then follows since $d\widetilde{m}(v)$ is a quasi-product measure of $d\mu^\iota_x(v_-)$, $d\mu_x(v_+)$ and dt.

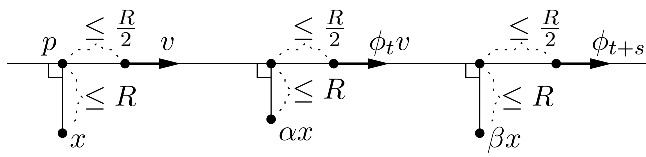
(e) This is immediate by the construction of m.

(f) Assume that $Q_{\Gamma,\,F}(\sigma) = +\infty$. Let $R > 0$ be large enough so that Lemma 5.2 (with $y = x$) holds. With $a \geq 0$ the maximum mass of an atom of μ^ι_x, also assume that R is large enough so that for every $\gamma \in \Gamma$, we have $\mu^\iota_x(\mathscr{O}^-_R(\gamma x, x)) \geq c_7 = \frac{\|\mu^\iota_x\|-a}{2}$, with

$\mathscr{O}_R^-(\gamma x, x)$ defined in Subsection 5.1. Let K be the compact subset $K(x,R)$ of $T^1\widetilde{M}$ defined in Subsection 5.1. The proof of Assertion (f) relies on the following two lemmas.

Lemma 5.6 *There exists $c_8 > 0$ such that for every sufficiently large $T > 0$*

$$\int_0^T \int_0^T \Big(\sum_{\alpha,\,\beta\in\Gamma} \widetilde{m}(K \cap \phi_{-t}\alpha K \cap \phi_{-s-t}\beta K)\Big)\, ds\, dt \le c_8 \Big(\sum_{\gamma\in\Gamma,\, d(x,\gamma x)\le T} e^{\int_x^{\gamma x}(\widetilde{F}-\sigma)} \Big)^2 .$$



Proof. By the definitions of m in Subsection 3.7, and of $K = K(x,R)$ and $L_R(x,\gamma x)$ in Subsection 5.1, for all $s,t \ge 0$ and $\alpha,\beta \in \Gamma$, we have (see the above picture)

$$\widetilde{m}(K \cap \phi_{-t}\alpha K \cap \phi_{-s-t}\beta K) \le R \int_{(\xi,\,\eta)\in L_R(x,\,\beta x)} \frac{d\mu_x^\iota(\xi)\; d\mu_x(\eta)}{D_{F-\sigma,\,x}(\xi,\eta)^2} .$$

Let $(\xi,\eta) \in L_R(x,\beta x)$ and p be the closest point to x on the geodesic line between ξ and η. By Equation (25) saying that

$$D_{F-\sigma,\,x}(\xi,\eta) = e^{-\frac{1}{2}(C_{F-\sigma,\,\eta}(x,\,p) + C_{F\circ\iota-\sigma,\,\xi}(x,\,p))} ,$$

by Lemma 3.4 (1) implying that

$$|\, C_{F-\sigma,\eta}(x,p)\,| \le c_1 e^R + R \sup_{\pi^{-1}(B(x,\,R))} |\widetilde{F} - \sigma|$$

and similarly by replacing F by $F\circ\iota$ and η by ξ, there exists a constant $c > 0$ such that

$$\frac{1}{c} \le D_{F-\sigma,\,x}(\xi,\eta) \le c\, . \tag{74}$$

Since $L_R(x,\beta x) \subset \partial_\infty\widetilde{M} \times \mathscr{O}_R^+(x,\beta x)$, and by Lemma 5.2, we hence have, for all $t,s \ge 0$,

$$\widetilde{m}(K \cap \phi_{-t}\alpha K \cap \phi_{-s-t}\beta K) \le c^2\; C\, R\; \|\mu_x^\iota\|\; e^{\int_x^{\beta x}(\widetilde{F}-\sigma)} .$$

By the triangle inequality (and since closest point projections on geodesic lines do not increase the distance), if $K \cap \phi_{-t}\alpha K \cap \phi_{-s-t}\beta K$ is nonempty (see the above picture), then

$$d(x,\alpha x) + d(\alpha x, \beta x) \le d(x,\beta x) + 4R\, .$$

Furthermore, by the definition of $K = K(x,R)$, we also have

$$d(x,\alpha x) - 3R \le t \le d(x,\alpha x) + R \quad\text{and}\quad d(\alpha x,\beta x) - 3R \le s \le d(\alpha x,\beta x) + R\, .$$

Since $d(\alpha x, [x,\beta x]) \le 2R$, we also observe by Lemma 3.2 that there exists a constant $c' \ge 0$ (depending only on R, on the Hölder constants of $\widetilde{F}$, on the bounds on the sectional curvature and on $\max_{\pi^{-1}(B(x,\,2R))} |\widetilde{F} - s|$) such that

$$\Big|\int_x^{\beta x} (\widetilde{F}-\sigma) - \int_x^{\alpha x} (\widetilde{F}-\sigma) - \int_{\alpha x}^{\beta x} (\widetilde{F}-\sigma)\Big| \le c'\, .$$

Therefore, since $m(K\cap\phi_{-t}\alpha K\cap\phi_{-s-t}\beta K)=0$ outside intervals of s and t of lengths $4R$, we have

$$\begin{aligned}
&\int_0^T\int_0^T\Big(\sum_{\alpha,\,\beta\in\Gamma}\widetilde{m}(K\cap\phi_{-t}\alpha K\cap\phi_{-s-t}\beta K)\Big)\,ds\,dt\\
&\le(4R)^2\sum_{\substack{\alpha\,\in\,\Gamma,\;d(x,\,\alpha x)\,\le\,T+3R\\ \beta\,\in\,\Gamma,\;d(\alpha x,\,\beta x)\,\le\,T+3R}} c^2\,C\,R\,\|\mu^\iota_x\|\,e^{c'}\,e^{\int_x^{\alpha x}(\widetilde{F}-\sigma)+\int_{\alpha x}^{\beta x}(\widetilde{F}-\sigma)}\\
&\le C_3\Big(\sum_{\gamma\in\Gamma,\,d(x,\gamma x)\le T+3R}e^{\int_x^{\gamma x}(\widetilde{F}-\sigma)}\Big)^2\,,
\end{aligned}$$

for some constant $C_3>0$. By Lemma 3.11 (1), the sum

$$\sum_{\gamma\in\Gamma,\,T<d(x,\gamma x)\le T+3R}e^{\int_x^{\gamma x}(\widetilde{F}-\sigma)}$$

is uniformly bounded as T tends to $+\infty$. Since the Poincaré series of (Γ,F) diverges at the point σ, the result follows. □

Lemma 5.7 *There exists $c_9>0$ such that for every sufficiently large $T>0$*

$$\int_0^T\Big(\sum_{\gamma\in\Gamma}\widetilde{m}(K\cap\phi_{-t}\gamma K)\Big)\,dt\ge c_9\sum_{\gamma\in\Gamma,\,d(x,\gamma x)\le T}e^{\int_x^{\gamma x}(\widetilde{F}-\sigma)}\,.$$

Proof. The proof is similar to the previous one. Let $\gamma\in\Gamma$ be such that $3R\le d(x,\gamma x)\le T-3R$. Then by the definition of $K=K(x,R)$, we have (see Subsection 5.1 for the definition of $v_{\xi,\,\eta,\,x}$)

$$\begin{aligned}
\int_0^T\widetilde{m}(K\cap\phi_{-t}\gamma K)\,dt&=\int_{(\xi,\,\eta)\in L_R(x,\,\gamma x)}\frac{d\mu^\iota_x(\xi)\;d\mu_x(\eta)}{D_{F-\sigma,\,x}(\xi,\eta)^2}\int_{-\frac{R}{2}}^{\frac{R}{2}}\int_0^T\mathbb{1}_{\gamma K}(\phi_{s+t}v_{\xi,\,\eta,\,x})\,dt\,ds\\
&=R^2\int_{(\xi,\eta)\in L_R(x,\gamma x)}\frac{d\mu^\iota_x(\xi)\;d\mu_x(\eta)}{D_{F-\sigma,\,x}(\xi,\eta)^2}\\
&\ge\frac{1}{c^2}\,R^2\,\mu^\iota_x(\mathscr{O}^-_R(\gamma x,x))\,\mu_x(\mathscr{O}^-_R(x,\gamma x))\,,
\end{aligned}$$

using the fact that $L_R(x,\gamma x)$ contains $\mathscr{O}^-_R(\gamma x,x)\times\mathscr{O}^-_R(x,\gamma x)$ (see Lemma 5.1) and the upper bound in Equation (74). By the assumption on R at the beginning of the proof of Assertion (f), and by the lower bound in Lemma 5.2, we have

$$\int_0^T\sum_{\gamma\in\Gamma}\widetilde{m}(K\cap\phi_{-t}\gamma K)\,dt\ge\sum_{\gamma\in\Gamma,\,3R\le d(x,\,\gamma x)\le T-3R}\frac{1}{c^2}\,R^2\,c_7\,\frac{1}{C}\,e^{\int_x^{\gamma x}(\widetilde{F}-\sigma)}\,.$$

We conclude as in the end of the proof of the previous Lemma 5.6. □

Now, to prove Assertion (f) from these two lemmas, let B be the image of K in $T^1M=\Gamma\backslash T^1\widetilde{M}$, let $A_t=B\cap\phi_{-t}B$, and let ν be the restriction of the measure m to the relatively compact set B. Then $\int_0^\infty\nu(A_t)\,dt=+\infty$ by Lemma 5.7 and the divergence of

the Poincaré series of (Γ, F) at the point σ. Furthermore, using simultaneously Lemma 5.6 and Lemma 5.7,

$$\int_0^T \int_0^T \nu(A_t \cap A_s)\, ds\, dt \leq 2 \int_0^T \int_0^T \nu(A_t \cap A_{s+t})\, ds\, dt \leq 2 \frac{c_8}{{c_9}^2} \Big(\int_0^T \nu(A_t)\, dt \Big)^2 .$$

By the appropriate version of the Borel-Cantelli Lemma (see for instance [Rob1, page 20]), we deduce that the set $\{v \in B :\ \int_0^\infty \mathbb{1}_{A_t}(v)\, dt = +\infty\}$ has positive measure with respect to ν. By Assertion (d) and the fact that $\widetilde{m}$ is a quasi-product measure, this implies that $\mu_x(\Lambda_c\Gamma) > 0$. Since $Q_{\Gamma,\,F}(\sigma)$ diverges if and only if $Q_{\Gamma,\,F\circ\iota}(\sigma)$ diverges, we also have $\mu^\iota_x(\Lambda_c\Gamma) > 0$.

(g) Observe first that, thanks to Mohsen's shadow lemma 3.10, the measure μ_x has the following "doubling property" (see for instance [Hei, page 3]): for every $R > 0$ large enough, there exists $C = C(x, R) > 0$ such that for every $\gamma \in \Gamma$, we have

$$\mu_x(\mathscr{O}_x B(\gamma x, 5R)) \leq C\, \mu_x(\mathscr{O}_x B(\gamma x, R)) .$$

Therefore, we deduce as in [Rob1, page 22-23] the following lemma. For every $R > 0$, let $\Lambda(R)$ be the set of $\xi \in \Lambda_c\Gamma$ that are limits of points in Γx at distance at most R of the geodesic ray starting from x and ending at ξ.

Lemma 5.8 *If $R > 0$ is large enough, for every μ_x-integrable maps $\varphi : \partial_\infty \widetilde{M} \to \mathbb{R}$, for μ_x-almost every $\xi \in \Lambda(R)$, when $d(x, \gamma x) \to +\infty$ and $\xi \in \mathscr{O}_x B(\gamma x, R)$, we have*

$$\frac{1}{\mu_x(\mathscr{O}_x B(\gamma x, R))} \int_{\mathscr{O}_x B(\gamma x, R)} \varphi\, d\mu_x \to \varphi(\xi) .$$

Since $\Lambda_c\Gamma = \bigcup_{n\in\mathbb{N}} \Lambda(n)$ and by the assumption of assertion (g), choose $R > 0$ large enough so that $\mu_x(\Lambda(R)) > 0$, and Lemma 3.10 with $K = \{x\}$ and Lemma 5.8 are satisfied. Let $B = \partial_\infty \widetilde{M} - \Lambda_c\Gamma$, let $\varphi = \mathbb{1}_B$ be the characteristic function of B and let $\xi \in \Lambda(R)$. Then $\varphi(\xi) = 0$ and there exists a sequence $(\gamma_n)_{n\in\mathbb{N}}$ in Γ such that $d(x, \gamma_n x) \to +\infty$ and $\xi \in \mathscr{O}_x B(\gamma_n x, R)$. By Lemma 5.8, we have

$$\lim_{n\to+\infty} \frac{\mu_x(B \cap \mathscr{O}_x B(\gamma_n x, R))}{\mu_x(\mathscr{O}_x B(\gamma_n x, R))} = 0 .$$

Using Equation (35) and Lemma 3.4 (2) to estimate the numerator, and Lemma 3.10 to estimate the denominator, there exists a constant $c > 0$ (depending on x, R, F) such that

$$\frac{1}{c}\, \mu_{\gamma_n x}(B \cap \mathscr{O}_x B(\gamma_n x, R)) \leq \frac{\mu_x(B \cap \mathscr{O}_x B(\gamma_n x, R))}{\mu_x(\mathscr{O}_x B(\gamma_n x, R))} \leq c\, \mu_{\gamma_n x}(B \cap \mathscr{O}_x B(\gamma_n x, R)) .$$

By Equation (34) and the invariance of B under Γ, we hence have

$$\lim_{n\to+\infty} \mu_x(B \cap \mathscr{O}_{\gamma_n^{-1} x} B(x, R)) = 0 .$$

Choosing arbitrarily large values of R, we may construct an increasing sequence $(U_k)_{k\in\mathbb{N}}$ of nonempty open subsets of $\partial_\infty X$, with $\lim_{k\to+\infty} \mu_x(B \cap U_k) = 0$, such that the diameter of the complement of $\partial_\infty X - U_k$ tends to 0. Up to extracting a subsequence, the closed subset $\partial_\infty X - U_k$ converges for the Hausdorff distance to a singleton $\{\zeta\}$. We have $\mu_x(B - \{\zeta\}) = 0$.

Since Γ is nonelementary, there exists $\gamma \in \Gamma$ such that $\gamma\zeta \neq \zeta$. By quasi-invariance of the measure μ_x and as Γ preserves B, we hence have $\mu_x(B) = 0$, as required.

(h) As in [Sul1, Rob1], the proof relies on well-known arguments of Hopf [Hop1]. The first part of it (before Lemma 5.9) could essentially be replaced by a reference to the nice paper [Cou3].

Fix a positive uniformly continuous map $\rho : T^1M \to \mathbb{R}$ with $\|\rho\|_{\mathbb{L}^1(m)} = 1$. A standard way of constructing such a ρ is as follows. Let d be the quotient distance on T^1M of the Riemannian distance on $T^1\widetilde{M}$, whose closed balls are compact. Fix $v_0 \in T^1M$ in the support of m. Let $f : [0, +\infty[\ \to\ ]0, +\infty[$ be the (positive, continuous, nonincreasing) piecewise linear map such that $f(n) = \frac{1}{2^n m(B(v_0,+1))}$. Note that f is integrable, since if $A_n = \{v \in T^1M\ :\ n \leq d(v_0, v) < n + 1\}$, then

$$\int_{T^1M} f\ dm = \sum_{n=0}^{\infty} \int_{A_n} f\ dm \leq \sum_{n=0}^{\infty} \frac{m(A_n)}{2^n\, m(B(v_0, n + 1))} \leq 2\ .$$

Take for ρ the integrable map $v \mapsto f(d(v_0, v))$ renormalised to have $\mathbb{L}^1$-norm 1. Note that ρ is continuous on the locally compact space T^1M and converges to 0 at infinity, hence is uniformly continuous.

As $(T^1M, (\phi_t)_{t\in\mathbb{R}}, m)$ is completely conservative (by the assumption of (h)), for m-almost every $v \in T^1M$, we have

$$\int_0^{\infty} \rho \circ \phi_{-t}(v)dt = \int_0^{\infty} \rho \circ \phi_t(v)dt = +\infty$$

(see for instance [Aar, Prop. 1.1.6, page 18] in the non invertible case). Let $\widetilde{\rho}\colon T^1\widetilde{M} \to \mathbb{R}$ be the lift of ρ by the canonical projection $T^1\widetilde{M} \to \Gamma\backslash T^1\widetilde{M} = T^1M$.

For every continuous map $h\colon T^1M \to \mathbb{R}$ with compact support, let $\widetilde{h}\colon T^1\widetilde{M} \to \mathbb{R}$ be its lift, and consider the following maps, whose domains of definition in $T^1\widetilde{M}$ are denoted by $\widetilde{D}^*$ and $\widetilde{D}_*$ respectively:

$$\widetilde{h}^*\colon v \mapsto \lim_{T\to+\infty} \frac{\int_0^T \widetilde{h} \circ \phi_t(v)\, dt}{\int_0^T \widetilde{\rho} \circ \phi_t(v)\, dt} \quad \text{and} \quad \widetilde{h}_*\colon v \mapsto \lim_{T\to+\infty} \frac{\int_0^T \widetilde{h} \circ \phi_{-t}(v)\, dt}{\int_0^T \widetilde{\rho} \circ \phi_{-t}(v)\, dt}\ .$$

Note that $\widetilde{D}^*$, $\widetilde{D}_*$, $\widetilde{h}^*$ and $\widetilde{h}_*$ are invariant under Γ and (m-almost everywhere) by the geodesic flow, since $\int_0^{\infty} \rho \circ \phi_{\pm t}(v)\, dt = +\infty$ for m-almost every v. Let h^* and h_* be the maps from the images of $\widetilde{D}^*$ and $\widetilde{D}_*$ in T^1M to $\mathbb{R}$ induced by $\widetilde{h}^*$ and $\widetilde{h}_*$ respectively. By the uniform continuity of h and ρ, the domains $\widetilde{D}^*$ and $\widetilde{D}_*$ are saturated respectively by the strong stable and strong unstable leaves of the geodesic flow, and furthermore $\widetilde{h}^*(v)$ depends only on v_+ and $\widetilde{h}_*(v)$ on v_-.

Let $\mathscr{I}$ be the σ-algebra of $(\phi_t)_{t\in\mathbb{R}}$-invariant Borel subsets of T^1M, let $\mathbb{P}$ be the probability measure ρm and let $E(\varphi\,|\,\mathscr{I})$ be the conditional expectation of $\varphi \in \mathbb{L}^1(\mathbb{P})$ with respect to $\mathscr{I}$: $E(\varphi\,|\,\mathscr{I})$ is $\mathbb{P}$-integrable and almost everywhere invariant under $(\phi_t)_{t\in\mathbb{R}}$ and for every bounded measurable map $\psi : T^1M \to \mathbb{R}$ which is almost everywhere invariant under $(\phi_t)_{t\in\mathbb{R}}$, we have

$$\int_{T^1M} \varphi\, \psi\ d\mathbb{P} = \int_{T^1M} E(\varphi\,|\,\mathscr{I})\, \psi\ d\mathbb{P}\ . \tag{75}$$

As $(T^1M, (\phi_t)_{t\in\mathbb{R}}, m)$ is completely conservative (by the assumption of (h)) and invertible, Hopf's ratio ergodic theorem (see for instance [Hop1], as well as [Zwe] for a short proof) implies that $\widetilde{D}^*$ and $\widetilde{D}_*$ have full measure with respect to $\widetilde{m}$, and that the maps h^*, h_* and $E_{\rho m}(\frac{h}{\rho}\,|\,\mathscr{I})$ are defined and coincide m-almost everywhere. Let us denote by ψ^* and ψ_* two bounded measurable maps on $\partial_\infty\widetilde{M}$ such that $\widetilde{h}^*(v) = \psi^*(v_+)$ and $\widetilde{h}_*(v) = \psi^*(v_-)$ for $\widetilde{m}$-almost every $v \in T^1\widetilde{M}$.

Lemma 5.9 *If $(T^1M, (\phi_t)_{t\in\mathbb{R}}, m)$ is completely conservative, then the product measure $\mu^\iota_x \otimes \mu_x$ is nonatomic.*

Proof. Let (ξ', ξ) be an atom of $\mu^\iota_x \otimes \mu_x$. Since Γ is nonelementary, there exists $\gamma \in \Gamma$ such that $\gamma\xi'$ is different from ξ and from the other fixed point of any hyperbolic element of Γ (if there is one) fixing ξ. Then by the quasi-invariance of μ^ι_x, the pair $(\gamma\xi', \xi)$ is another atom of $\mu^\iota_x \otimes \mu_x$. Let $\epsilon > 0$ be small enough. Let $v \in T^1\widetilde{M}$ be such that $v_- = \gamma\xi'$ and $v_+ = \xi$. Then the image of $\{\phi_t v \;:\; t \in [-\epsilon, \epsilon]\}$ in T^1M is a measurable subset of positive measure with respect to m which is a wandering set for the geodesic flow, since $\{\gamma\xi', \xi\}$ is not the set of fixed points of a hyperbolic element. $\square$

It follows from this lemma, as in the proof of Assertion (c), that the diagonal of $\partial_\infty\widetilde{M} \times \partial_\infty\widetilde{M}$ has measure 0 for the product measure $\mu^\iota_x \otimes \mu_x$. Consider the probability measures $\overline{\mu}^\iota = \frac{\mu^\iota_x}{\|\mu^\iota_x\|}$ and $\overline{\mu} = \frac{\mu_x}{\|\mu_x\|}$. Since $\widetilde{m}$ is equivalent to the product measure $d\mu^\iota_x\, d\mu_x\, dt$ on $\partial^2_\infty\widetilde{M} \times \mathbb{R}$, we have, using Fubini's theorem, the Hopf parametrisation of Remark (2) of Subsection 3.7 (with $x_0 = x$), and the fact that h_* and h^* coincide almost everywhere for the measure $d\overline{\mu}^\iota(v_-)\, d\overline{\mu}(v_+)\, dt$,

$$\begin{aligned}
\int_{\partial_\infty\widetilde{M}} \psi^*(\eta)^2\, d\overline{\mu}(\eta) &= \int_0^1 \int_{\partial_\infty\widetilde{M}\times\partial_\infty\widetilde{M}} \psi^*(\eta)^2\, d\overline{\mu}^\iota(\xi)\, d\overline{\mu}(\eta)\, dt \\
&= \int_{\partial^2_\infty\widetilde{M}\times\mathbb{R}} \mathbb{1}_{[0,1]}(t)\ \psi^*(\eta)^2\, d\overline{\mu}^\iota(\xi)\, d\overline{\mu}(\eta)\, dt \\
&= \int_{T^1\widetilde{M}} \mathbb{1}_{[0,1]}(\beta_{v_-}(\pi(v), x))\ h^*(v)^2\, d\overline{\mu}^\iota(v_-)\, d\overline{\mu}(v_+)\, dt \\
&= \int_{T^1\widetilde{M}} \mathbb{1}_{[0,1]}(\beta_{v_-}(\pi(v), x))\ h_*(v)h^*(v)\, d\overline{\mu}^\iota(v_-)\, d\overline{\mu}(v_+)\, dt \\
&= \int_0^1 \int_{\partial_\infty\widetilde{M}\times\partial_\infty\widetilde{M}} \psi_*(\xi)\psi^*(\eta)\, d\overline{\mu}^\iota(\xi)\, d\overline{\mu}(\eta)\, dt \\
&= \int_{\partial_\infty\widetilde{M}} \psi_*(\xi)\, d\overline{\mu}^\iota(\xi) \int_{\partial_\infty\widetilde{M}} \psi^*(\eta)\, d\overline{\mu}(\eta) \\
&= \int_{\partial_\infty\widetilde{M}\times\partial_\infty\widetilde{M}\times[0,1]} \psi_*(\xi)\, d\overline{\mu}^\iota(\xi)\, d\overline{\mu}(\eta)\, dt \int_{\partial_\infty\widetilde{M}} \psi^*(\eta)\, d\overline{\mu}(\eta) \\
&= \int_{\partial_\infty\widetilde{M}\times\partial_\infty\widetilde{M}\times[0,1]} \psi^*(\eta)\, d\overline{\mu}^\iota(\xi)\, d\overline{\mu}(\eta)\, dt \int_{\partial_\infty\widetilde{M}} \psi^*(\eta)\, d\overline{\mu}(\eta) \\
&= \Big(\int_{\partial_\infty\widetilde{M}} \psi^*(\eta)\, d\overline{\mu}(\eta)\Big)^2 .
\end{aligned}$$

By the equality case in the Cauchy-Schwarz inequality, we hence have that ψ^* is constant μ_x-almost everywhere. By Fubini's theorem and the fact that $\widetilde{m}$ is equivalent to the

product measure $d\mu^\iota_x\, d\mu_x\, dt$ on $\partial^2_\infty \widetilde{M} \times \mathbb{R}$, the preimage of a μ_x-measure zero subset of $\partial_\infty \widetilde{M}$ by $v \mapsto v_+$ is a $\widetilde{m}$-measure zero subset of $T^1\widetilde{M}$. We hence have that $\widetilde{h}^*$ is $\widetilde{m}$-almost everywhere constant.

Since the map $E(\frac{h}{\rho}\,|\,\mathscr{I})$ is m-almost everywhere constant, it is $\mathbb{P}$-almost everywhere equal to $\int_{T^1M} \frac{h}{\rho}\; d\mathbb{P} = \int_{T^1M} h\; dm$ by Equation (75). Let $\psi \in \mathbb{L}^1(\mathbb{P})$ be invariant under the geodesic flow. From the density in $\mathbb{L}^1(\mathbb{P})$ of the subspace of continuous maps with compact support, we deduce that $\int_{T^1M} \psi^2\; d\mathbb{P} = (\int_{T^1M} \psi\; d\mathbb{P})^2$. By the equality case in the Cauchy-Schwarz inequality, this implies that ψ is almost everywhere constant. Since $\mathbb{P}$ and m are equivalent, this says that $(T^1M, (\phi_t)_{t\in\mathbb{R}}, m)$ is ergodic. $\Box$

Proofs of Theorem 5.3 and Theorem 5.4. We use the numbering of the assertions (1)-(5) of Theorem 5.3, (i)-(v) of Theorem 5.4 and (a)-(g) of Proposition 5.5. Up to replacing F by $F + c$ (and σ by $\sigma + c$) where c is a big enough constant, which does not change the validity of the assertions of Theorem 5.3 and Theorem 5.4, we may assume that $\sigma > 0$.

The proof of Theorem 5.3 follows from the following implications:

- (1), (2) and (3) are equivalent, by (a) and (f).
- (5) implies (2), by (d).
- (2) and (3) imply (4), by (b) and (c).
- (2) and (3) and (4) imply (5) by (e) and (d).
- Finally, assume that Assertion (4) holds, let us prove that Assertion (1) holds. Since $(\partial^2_\infty \widetilde{M}, \Gamma, (\mu^\iota_x \otimes \mu_x)_{|\partial^2_\infty \widetilde{M}})$ is non-ergodic, $(T^1M, (\phi_t)_{t\in\mathbb{R}}, m)$ is non-ergodic by (e). By the contrapositive of (h), $(T^1M, (\phi_t)_{t\in\mathbb{R}}, m)$ is not completely conservative. By the contrapositive of (d), we have $\mu_x({}^c\Lambda_c\Gamma) > 0$ or $\mu^\iota_x({}^c\Lambda_c\Gamma) > 0$. By the contrapositive of (g), which is also valid upon replacement of μ_x by μ^ι_x, we have $\mu_x(\Lambda_c\Gamma) = 0$ or $\mu^\iota_x(\Lambda_c\Gamma) = 0$. By the contrapositive of (f), this proves (1).

The proof of Theorem 5.4 follows from the following implications:

- (ii) and (iii) are equivalent, by (g) which is also valid upon replacement of μ_x by μ^ι_x, since (2) and (3) are.
- (i) implies (ii), by (f) and (g).
- (ii) and (iii) imply (v), by (d) and (h).
- (v) implies (i), by (d) and the contrapositive of (a).
- (v) implies (iv), by (e) and (c).
- (iv) implies (i), by (c) and the contrapositives of (b) and (a). $\Box$

5.3 Uniqueness of Patterson densities and of Gibbs states

Let $\widetilde{M}, \Gamma, \widetilde{F}$ be as in the beginning of Chapter 2.

The first result in this subsection, generalising the case $F = 0$ due to Sullivan [Sul2], gives a useful criterion for being of divergence type.

Corollary 5.10 *Assume that $\delta_{\Gamma,\,F} < +\infty$ and that there exists a Patterson density $(\mu_x)_{x\in X}$ for (Γ, F) of dimension $\sigma \geq \delta_{\Gamma,\,F}$ such that $\mu_x(\Lambda_c\Gamma) > 0$. Then $\sigma = \delta_{\Gamma,\,F}$ and (Γ, F) is of divergence type.*

Proof. This follows immediately from Theorem 5.3. $\Box$

In particular, if Γ is convex-cocompact, then Patterson's construction recalled in the proof of Proposition 3.9 shows that (Γ, F) is of divergence type.

Proposition 5.11 *Let $\sigma \in \mathbb{R}$.*

(1) If Γ is convex-cocompact, then there exists a Patterson density for (Γ, F) of dimension σ with support equal to $\Lambda\Gamma$ if and only if $\sigma = \delta_{\Gamma,\,F}$.

(2) If Γ is not convex-cocompact, then there exists a Patterson density for (Γ, F) of dimension σ with support equal to $\Lambda\Gamma$ if and only if $\sigma \geq \delta_{\Gamma,\,F}$.

Proof. By Corollary 3.11, we always have $\sigma \geq \delta_{\Gamma,\,F}$ if there exists a Patterson density for (Γ, F) of dimension σ. Since $\Lambda\Gamma = \Lambda_c\Gamma$ when Γ is convex-cocompact, the first claim follows from Patterson's construction (see Proposition 3.9) for the existence part when $\sigma = \delta_{\Gamma,\,F}$, and from the implication (1) implies (2) of Theorem 5.3 for the converse statement.

To prove the second claim, by Corollary 3.11 and Proposition 3.9, we only have to prove that given $\sigma > \delta_{\Gamma,\,F}$, there exists a Patterson density for (Γ, F) of dimension σ with support equal to $\Lambda\Gamma$. Since Γ is not convex-cocompact, there exists a sequence $(y_i)_{i\in\mathbb{N}}$ in $\mathscr{C}\Lambda\Gamma$ such that $\lim_{i\to+\infty} d(y_i, \Gamma x_0) = +\infty$. For every $x \in \widetilde{M}$, consider the nonzero (since $\sigma > \delta_{\Gamma,\,F}$) measure

$$\mu_{x,\,i} = \frac{\sum_{\gamma\in\Gamma} e^{\int_x^{\gamma y_i}(\widetilde{F}-\sigma)}\mathscr{D}_{\gamma y_i}}{\sum_{\gamma\in\Gamma} e^{\int_{x_0}^{\gamma y_i}(\widetilde{F}-\sigma)}}\;,$$

whose support is contained in $\mathscr{C}\Lambda\Gamma$, and whose total mass is bounded from above and below by a positive constant independent of i by Lemma 3.2. The family $(\mu_{x,\,i})_{x\in\widetilde{M}}$ has an accumulation point (for the product topology of the weak star topologies) $(\mu_x)_{x\in\widetilde{M}}$ of finite nonzero measures with support contained in (hence equal to) $\Lambda\Gamma$, which is easily proven to be a Patterson density for (Γ, F) of dimension σ. □

The divergence type assumption has several nice consequences.

Corollary 5.12 *Assume in this corollary that $\delta_{\Gamma,\,F} < +\infty$ and that (Γ, F) is of divergence type. Then there exists a Patterson density $(\mu_{F,\,x})_{x\in\widetilde{M}}$ of dimension $\delta_{\Gamma,\,F}$ for (Γ, F), which is unique up to a scalar multiple. If $\mathscr{D}_z$ is the unit Dirac mass at a point $z \in \widetilde{M}$, then for every $x \in \widetilde{M}$, we have*

$$\frac{\mu_{F,\,x}}{\|\mu_{F,\,x}\|} = \lim_{s\to\delta_{\Gamma,\,F}{}^+} \frac{1}{Q_{\Gamma,\,F,\,x,\,x}(s)} \sum_{\gamma\in\Gamma} e^{\int_x^{\gamma x}(\widetilde{F}-s)}\,\mathscr{D}_{\gamma x}\;. \tag{76}$$

The measures $\mu_{F,\,x}$ have no atom ξ such that $\int_x^\xi(\widetilde{F} - \delta_{\Gamma,\,F}) = -\infty$, and their support is $\Lambda\Gamma$. Furthermore, $\mu_{F,\,x}(\Lambda\Gamma - \Lambda_{\rm Myr}\Gamma) = 0$, and in particular $\mu_{F,\,x}(\Lambda\Gamma - \Lambda_c\Gamma) = 0$.

Equation (76) says that Patterson's construction of the Patterson densities (see the proof of Proposition 3.9) does not depend on choices of accumulation points (and follows from the uniqueness property). This is due to Patterson when $F = 0$. In the context of Gibbs measures (with the multiplicative convention, as explained in the beginning of Chapter 3), this has been first proved by Ledrappier [Led2].

Proof. The existence has been seen in Proposition 3.9. The uniqueness is classical (see [Sul1]): If $(\mu'_{F,\,x})_{x\in\widetilde{M}}$ is another Patterson density of dimension $\delta_{\Gamma,\,F}$ for (Γ, F), then so is

$(\mu_{F,x}+\mu'_{F,x})_{x\in\widetilde{M}}$. For any fixed $x\in\widetilde{M}$, let $\mu=\mu_{F,x}$, $\mu'=\mu'_{F,x}$ and $\nu=\mu+\mu'$. Note that μ is absolutely continuous with respect to ν, and that the Radon-Nikodym derivative $\frac{d\nu}{d\mu}$ is Γ-invariant and measurable, hence is μ-almost everywhere equal to a constant $\lambda>0$ by ergodicity (see Theorem 5.4). Therefore $\nu=\lambda\mu$ and $\mu'=(\lambda-1)\mu$.

The fact that the support of $\mu_{F,x}$ is $\Lambda\Gamma$ follows from its construction (see Proposition 3.9).

Assume for a contradiction that there exists an atom $\xi\in\partial_\infty\widetilde{M}$ of $\mu_{F,x}$ such that $\int_x^\xi(\widetilde{F}-\delta_{\Gamma,F})=-\infty$. Since $\mu_{F,x}$ gives full measure to the conical limit set (see Theorem 5.4), we have $\xi\in\Lambda_c\Gamma$. But then by Mohsen's shadow lemma 3.10 with $K=\{x\}$, there exist $C>0, R>0$ and a sequence of elements $(\gamma_i)_{i\in\mathbb{N}}$ in Γ such that $(\gamma_i x)_{i\in\mathbb{N}}$ converges to ξ while staying at bounded distance from the geodesic ray $[x,\xi[$, such that $\xi\in\mathscr{O}_xB(\gamma_i x,R)$, and such that $\mu_{F,x}(\mathscr{O}_xB(\gamma_i x,R))\leq C\quad e^{\int_x^{\gamma_i x}(\widetilde{F}-\delta_{\Gamma,F})}$. Since $\widetilde{F}$ is Γ-invariant, the map $\widetilde{F}-\delta_{\Gamma,F}$ is bounded on the balls of center $\gamma_i x$ and given radius. Hence by Lemma 3.2, there exists $C'>0$ such that, if p_i is the closest point to $\gamma_i x$ on $[x,\xi[$, we have $\mu_{F,x}(\{\xi\})\leq C'\,e^{\int_x^{p_i}(\widetilde{F}-\delta_{\Gamma,F})}$. Therefore $\mu_{F,x}(\{\xi\})=0$, a contradiction.

The proof that $\mu_x(\Lambda\Gamma-\Lambda_{\rm Myr}\Gamma)=0$ is the same one as when $F=0$ in [Rob1] (generalising [Tuk]): Let $\mathscr{B}$ be a countable basis of the topology of the support of the Gibbs measure $\widetilde{m}_F$ associated to $(\mu_{F\circ\iota,x})_{x\in\widetilde{M}}$ and $(\mu_{F,x})_{x\in\widetilde{M}}$, where $(\mu_{F\circ\iota,x})_{x\in\widetilde{M}}$ is any Patterson density of dimension $\delta_{\Gamma,F}=\delta_{\Gamma\circ\iota,F}$ for $(\Gamma,F\circ\iota)$, which exists by Proposition 3.9. For every nonempty $U\in\mathscr{B}$, let $\mathscr{M}(U)$ be the set of $v\in T^1\widetilde{M}$ for which there exist sequences $(\gamma_i)_{i\in\mathbb{N}}$ in Γ and $(t_i)_{i\in\mathbb{N}}$ in $\mathbb{R}$ tending to $+\infty$ such that $\phi_{t_n}(\gamma_n v)\in U$. Then $\mathscr{M}(U)$ is a Γ-invariant measurable subset of $T^1\widetilde{M}$. Since $(T^1M,(\phi_t)_{t\in\mathbb{R}},m_F)$ is completely conservative (see Theorem 5.4), the subset U is contained in $\mathscr{M}(U)$. Since $\widetilde{m}_F(U)>0$ and by ergodicity (see Theorem 5.4), the set $\mathscr{M}(U)$ has full measure. Since $\Lambda_{\rm Myr}\Gamma=\{v_+\;:\;v\in\bigcap_{U\in\mathscr{B}}\mathscr{M}(U)\}$ by definition (see Chapter 2), and since by the quasi-product structure of $\widetilde{m}_F$, the preimage by $v\mapsto v_+$ of a subset of positive measure has positive measure, we have that $\mu_{F,x}(\Lambda\Gamma-\Lambda_{\rm Myr}\Gamma)=0$. □

Corollary 5.13 *Let $\sigma\in\mathbb{R}$. There exists a Patterson density of dimension σ for (Γ,F) whose support contains strictly $\Lambda\Gamma$ if and only if $\Lambda\Gamma\neq\partial_\infty\widetilde{M}$ and the Poincaré series $\sum_{\gamma\in\Gamma}e^{\int_x^{\gamma x}(\widetilde{F}-\sigma)}$ converges.*

Proof. Assume first that $\Lambda\Gamma\neq\partial_\infty\widetilde{M}$ and that the Poincaré series $\sum_{\gamma\in\Gamma}e^{\int_x^{\gamma x}(\widetilde{F}-\sigma)}$ converges. Let $\xi\in\partial_\infty\widetilde{M}-\Lambda\Gamma$ and let $(y_i)_{i\in\mathbb{N}}$ be a sequence in $\widetilde{M}$ converging to ξ. For every $x\in\widetilde{M}$, consider the nonzero measure

$$\mu_{x,i}=\frac{\sum_{\gamma\in\Gamma}e^{\int_x^{\gamma y_i}(\widetilde{F}-\sigma)}\mathscr{D}_{\gamma y_i}}{\sum_{\gamma\in\Gamma}e^{\int_{x_0}^{\gamma y_i}(\widetilde{F}-\sigma)}}\;.$$

As in the proof of Proposition 5.11 (2), the family $(\mu_{x,i})_{x\in\widetilde{M}}$ has an accumulation point $(\mu_x)_{x\in\widetilde{M}}$, which is a Patterson density for (Γ,F) of dimension σ. Its support is $\overline{\Gamma\xi}=\Gamma\xi\cup\Lambda\Gamma$, hence contains strictly $\Lambda\Gamma$.

Conversely, assume that there exists a Patterson density $(\mu_x)_{x\in\widetilde{M}}$ of dimension σ for (Γ,F) whose support contains strictly $\Lambda\Gamma$. In particular, $\Lambda\Gamma\neq\partial_\infty\widetilde{M}$. We have $\delta_{\Gamma,F}\leq\sigma$ by Corollary 3.11, and in particular $\delta_{\Gamma,F}<+\infty$. If the Poincaré series $\sum_{\gamma\in\Gamma}e^{\int_x^{\gamma x}(\widetilde{F}-\sigma)}$

diverges, then $\sigma = \delta_{\Gamma,\,F}$, and (Γ, F) is of divergence type. Hence by Corollary 5.12, the support of μ_x is equal to $\Lambda\Gamma$, a contradiction. $\square$

Corollary 5.14 *Let $\sigma \in \mathbb{R}$. If there exists a Gibbs measure of dimension σ for (Γ, F) which is finite on T^1M, then $\sigma = \delta_{\Gamma,\,F}$, the pair (Γ, F) is of divergence type, there exists a unique (up to a scalar multiple) Gibbs measure m_F with potential F on T^1M, its support is $\Omega\Gamma$, the geodesic flow $(\phi_t)_{t\in\mathbb{R}}$ on T^1M is ergodic and completely conservative with respect to m_F, and the union of the set of fixed points in $T^1\widetilde{M}$ of the elements of Γ which do not pointwise fix $\Lambda\Gamma$ has measure 0 for $\widetilde{m}_F$.*

It follows in particular from the first statement that if Γ is convex-cocompact, then there exists no Patterson density of dimension $\sigma > \delta_{\Gamma,\,F}$ (see also Proposition 5.11).

Proof. Let m be a finite Gibbs measure of dimension σ for (Γ, F). Up to adding a constant to F, we may assume that $\sigma > 0$. Since $(\phi_t)_{t\in\mathbb{R}}$ preserves m, Poincaré's recurrence theorem (see for instance [Kre, page 16]) says that $(T^1M, (\phi_t)_{t\in\mathbb{R}}, m)$ is completely conservative. Hence by Theorem 5.3, the Poincaré series of (Γ, F) diverges at σ. Since there are no Patterson densities of dimension strictly less than $\delta_{\Gamma,\,F}$ (by Corollary 3.11 (2)), and since $Q_{\Gamma,\,F}(s) < +\infty$ if $\sigma > \delta_{\Gamma,\,F}$, this implies that $\sigma = \delta_{\Gamma,\,F}$. Hence (Γ, F) is of divergence type. By Theorem 5.4, $(\phi_t)_{t\in\mathbb{R}}$ is ergodic for m. The uniqueness property of the Gibbs measure m_F of potential F then follows by its construction from Corollary 5.12. Since the (topological) nonwandering set $\Omega\Gamma$ (see Subsection 2.4) is the image in T^1M of the set of the elements $v \in T^1\widetilde{M}$ such that $v_-, v_+ \in \Lambda\Gamma$, the claim on the support of m_F follows also from Corollary 5.12.

The union A of the fixed points sets in $T^1\widetilde{M}$ of the elements of Γ which do not pointwise fix $\Lambda\Gamma$ is invariant under Γ and by the geodesic flow. It is closed by discreteness, and properly contained in $\widetilde{\Omega}\Gamma$, by the remark following Lemma 2.1. Note that $\widetilde{\Omega}\Gamma$ is the support of $\widetilde{m}_F$ (see Subsection 3.7). Hence by ergodicity, the measure of A with respect to $\widetilde{m}_F$ is 0. $\square$

Remark. If $\delta_{\Gamma,\,F} < +\infty$ and (Γ, F) is of divergence type (which implies that $(\Gamma, F\circ\iota)$ also is, by Equation (16) in Lemma 3.3), it follows from the uniqueness statement in Corollary 5.12 that if we normalise the Patterson density families for (Γ, F) and $(\Gamma, F\circ\iota)$ given by Corollary 5.12 to have total mass 1 for a given point $x_0 \in \widetilde{M}$, and if we consider the associated Gibbs measures $\widetilde{m}_F$ and $\widetilde{m}_{F\circ\iota}$, then

- $(\mu_{F,\,x})_{x\in\widetilde{M}} = (\mu_{F+s,\,x})_{x\in\widetilde{M}}$ for every s in $\mathbb{R}$ (by Equation (15));
- $\iota_*\widetilde{m}_F = \widetilde{m}_{F\circ\iota}$ and $\iota_* m_F = m_{F\circ\iota}$ (by Equation (26)); in particular, when $\widetilde{F}$ is reversible, we have $(\mu_{F\circ\iota,\,x})_{x\in\widetilde{M}} = (\mu_{F,\,x})_{x\in\widetilde{M}}$ and $\iota_*\widetilde{m}_F = \widetilde{m}_F$.
- for every $s \in \mathbb{R}$, if F' is a potential cohomologous to F, then $m_{F'+s} = m_F$ (by the remarks at the end of Subsection 3.6 and Subsection 3.7).

In the next chapters, we will often assume that there exists a finite Gibbs measure on T^1M with potential F. It is then ergodic and unique up to normalisation, by Corollary 5.14. In particular, T^1M, endowed with a given potential F, does not carry simultaneously a finite Gibbs measure and an infinite one.

6 Thermodynamic formalism and equilibrium states

Let $\widetilde{M}, \Gamma, \widetilde{F}$ be as in the beginning of Chapter 2: $\widetilde{M}$ is a complete simply connected Riemannian manifold, with dimension at least 2 and pinched sectional curvature at most -1; Γ is a nonelementary discrete group of isometries of $\widetilde{M}$; and $\widetilde{F} : T^1\widetilde{M} \to \mathbb{R}$ is a Hölder-continuous Γ-invariant map. In this chapter, we denote by $\phi = (\phi_t)_{t\in\mathbb{R}}$ the geodesic flow, both on $T^1\widetilde{M}$ and on $T^1M = \Gamma\backslash T^1\widetilde{M}$.

Given a measurable map $H : T^1M \to \mathbb{R}$, we denote by

$$H_- : v \mapsto \max\{0, -H(v)\}$$

the *negative part* of H. Note that for every positive Borel measure m on T^1M for which the negative part of H is integrable, the integral $\int_{T^1M} H\, dm$ is well defined in $\mathbb{R} \cup \{+\infty\}$ as

$$\int_{T^1M} H\, dm = \int_{T^1M} \max\{0, H\}\, dm - \int_{T^1M} H_-\, dm\ .$$

Note that if m, m_1, m_2 are positive Borel measures on T^1M and if $m = tm_1 + (1-t)m_2$ for some $t \in\,]0,1[\,$, then the negative part of F is integrable with respect to m if and only if it is integrable with respect to both m_1 and m_2.

Let $\mathscr{M}_F(T^1M)$ be the space of ϕ-invariant Borel probability measures m on T^1M such that the negative part of F is m-integrable, endowed with the weak star topology. Note that $\mathscr{M}_F(T^1M)$ is convex, but in general not closed in the convex weak star compact space of all ϕ-invariant Borel probability measures on T^1M.

For every $m \in \mathscr{M}_F(T^1M)$, the *(metric) pressure* of the potential F with respect to m is the element of $\mathbb{R} \cup \{+\infty\}$ defined by

$$P_{\Gamma,\,F}(m) = h_m(\phi) + \int_{T^1M} F\, dm\ ,$$

where $h_m(\phi)$ is the (metric) entropy of the geodesic flow ϕ with respect to m (see [KaH] or Subsection 6.1 for the properties we will need on the entropy). The *(topological) pressure* of the potential F is the upper bound of its metric pressures, that is, the element of $\mathbb{R}\cup\{+\infty\}$ defined by

$$P(\Gamma, F) = \sup_{m\in\mathscr{M}_F(T^1M)} P_{\Gamma,F}(m)\ .$$

An element $m \in \mathscr{M}_F(T^1M)$ realizing the upper bound, that is, such that $P(\Gamma, F) = P_{\Gamma,F}(m)$, is called an *equilibrium state* for (Γ, F). Using the convexity properties of $\mathscr{M}_F(T^1M)$ and of the metric entropy, and the ergodic decomposition of probability measures invariant under the geodesic flow, the pressure $P(\Gamma, F)$ of F is also the upper bound of the metric pressures of F with respect to the ϕ-ergodic elements in $\mathscr{M}_F(T^1M)$ (whose support is contained in the (topological) nonwandering set $\Omega\Gamma$). Furthermore, by the Krein-Milman theorem, any equilibrium state is ϕ-ergodic.

Note that for every $m \in \mathscr{M}_F(T^1M)$, we have $\iota_* m \in \mathscr{M}_{F\circ\iota}(T^1M)$ and $P_{\Gamma,\,F\circ\iota}(\iota_* m) = P_{\Gamma,\,F}(m)$. Hence

$$P(\Gamma, F) = P(\Gamma, F\circ\iota)\ .$$

The aim of this Chapter 6 is, following the case $F \equiv 0$ in [OtP], to prove the following result, saying in particular that a finite Gibbs state, once renormalised to be a probability measure, is the unique equilibrium state of a given bounded potential.

Theorem 6.1 *Let $\widetilde{M}$ be a complete simply connected Riemannian manifold, with dimension at least 2 and pinched negative sectional curvature. Let Γ be a nonelementary discrete group of isometries of $\widetilde{M}$. Let $\widetilde{F} : T^1\widetilde{M} \to \mathbb{R}$ be a Hölder-continuous Γ-invariant map such that $\delta_{\Gamma,\, F} < +\infty$.*

(1) We have
$$P(\Gamma, F) \leq \delta_{\Gamma,\, F}\ . \tag{77}$$

(2) If F is bounded from below on the (topological) nonwandering set $\Omega\Gamma$, then
$$P(\Gamma, F) = \delta_{\Gamma,\, F}\ . \tag{78}$$

(3) If there exists a finite Gibbs measure m_F for (Γ, F) such that the negative part of F is m_F-integrable, then we also have $P(\Gamma, F) = \delta_{\Gamma,\, F}$ and $m^F = \frac{m_F}{\|m_F\|}$ is the unique equilibrium state for (Γ, F). Otherwise, there exists no equilibrium state for (Γ, F).

The hypothesis in (2) that F has a finite lower bound may be weakened: We will see in the proof of Lemma 6.7 (towards its end) that it is enough to assume that the lower bound of F on a ball of radius N does not decrease too fast to $-\infty$ as $N \to +\infty$.

It follows in particular from Equation (77) that the topological pressure $P(\Gamma, F)$ is finite if the critical exponent $\delta_{\Gamma,\, F}$ is finite.

Remark. When Γ is torsion-free and convex-cocompact, Theorem 6.1 has been proved in [Sch3]. In this case, the critical exponent δ_Γ of Γ is the topological entropy h of the geodesic flow of M restricted to the (topological) nonwandering set $\Omega\Gamma$ (see [Mann]), and $\delta_{\Gamma,\, F}$ is the topological pressure $P(F)$ of the potential function $F : T^1M \to \mathbb{R}$ induced by $\widetilde{F}$ (see [Rue2]), that is

$$P(F) = \max_{m \in \mathscr{P}}\ \Big(\, h_m(\phi) + \int_{T^1M} F\, dm\, \Big)$$

where $\mathscr{P}$ is the set of all Borel probability measures on T^1M invariant under the geodesic flow. Note that the reference [Sch3] uses the multiplicative convention for the Poincaré series as mentioned in the beginning of Chapter 3, but the uniqueness property ensures that the two constructions of the Gibbs measures coincide in this case. Note that any Gibbs measure is finite in the convex-cocompact case, and uniqueness follows from general argument of [BoR], hence the proof (using the Shadow lemma and entropy computations of [Kai]) is much easier and shorter in this particular case.

An immediate consequence of the inequality $P(\Gamma, F) \leq \delta_{\Gamma,\, F}$ is that the periods of the normalised potential are nonpositive.

Corollary 6.2 *For every periodic orbit g of the geodesic flow on T^1M, we have*
$$\int_g (F - \delta_{\Gamma,\, F})\ \leq 0\ .$$

Proof. Let $\mathscr{L}_g$ be the Lebesgue measure along a periodic orbit Lebesgue measure along a periodic g with length $\ell(g)$ (see the beginning of Subsection 4.1). The measure $\frac{\mathscr{L}_g}{\ell(g)}$ is a

probability measure (with compact support) on T^1M invariant by the geodesic flow, whose metric entropy is 0 (since rotations of the circle have entropy zero). Hence

$$\delta_{\Gamma,\,F} \geq P(\Gamma, F) \geq P_{\Gamma,F}(m) = \int_{T^1M} F\;dm = \frac{\mathscr{L}_g(F)}{\ell(g)}\;,$$

which proves the result. □

In particular, for every measure μ on T^1M with compact support which belongs to the weak-star closure of the set of linear combinations (with nonpositive coefficients) of the Lebesgue measures along periodic orbits, we have

$$\int_{T^1M} (F - \delta_{\Gamma,\,F})\;d\mu\;\leq 0\;.$$

6.1 Measurable partitions and entropy

We refer to [Roh, Par1, KaH] for the generalities (at the strict minimum for what follows) on measurable partitions and entropy recalled in this subsection, that the knowledgeable reader may skip.

Recall that a *partition* ζ of a set X is a set of pairwise disjoint nonempty subsets of X whose union is X. We will denote by $\pi_\zeta : X \to \zeta$ the canonical projection map which maps every $x \in X$ to the (unique) element $z = \zeta(x)$ of ζ containing x. Given two partitions ζ and ζ' of X, we define their *refinement* as the partition

$$\zeta \vee \zeta' = \{z \cap z'\;:\; z \in \zeta, z' \in \zeta', z \cap z' \neq \emptyset\}\;,$$

and similarly for finitely many partitions.

Recall that two probability spaces with complete σ-algebras are *isomorphic* if there exists a measure-preserving bimeasurable bijection between full measure subsets of them. A probability space is *standard* if its σ-algebra is complete and if it is isomorphic to the probability space with underlying set the disjoint union of a compact interval with a countable set, inducing on the interval the Lebesgue σ-algebra and the Lebesgue measure and on the countable set (which is measurable) the discrete σ-algebra.

Fix a standard probability space $(X, \mathscr{A}, m)$. The following result that we state without proof is well known (see for instance [OtP, Lemme 7]).

Lemma 6.3 *If $(\phi_t)_{t\in\mathbb{R}}$ is a m-ergodic measure preserving flow on X, then there exists a countable subset D of $\mathbb{R}$ such that ϕ_τ is m-ergodic for every $\tau \in \mathbb{R} - D$.*

For every partition ζ of the set X, denote by $\widehat{\zeta}$ the sub-σ-algebra of elements of $\mathscr{A}$ which are unions of elements of ζ. Recall that ζ is *m-measurable* (or *measurable* if $(\mathscr{A}, m)$ is implicit) if there exist a full measure subset Y in X and a sequence $(A_n)_{n\in\mathbb{N}}$ in $\widehat{\zeta}$, satisfying the following separation property: for all $z \neq z'$ in ζ, there exists $n \in \mathbb{N}$ such that either we have $z \cap Y \subset A_n$ and $z' \cap Y \subset X - A_n$ or we have $z' \cap Y \subset A_n$ and $z \cap Y \subset X - A_n$. For instance, a finite partition of X is measurable if and only if each element of the partition is measurable. Given two m-measurable partitions ζ, ζ', we write $\zeta \preceq \zeta'$ (the conventions differ amongst the references) if $\zeta'(x) \subset \zeta(x)$ for m-almost every $x \in X$, and we say that ζ and ζ' are *m-equivalent* if $\zeta \preceq \zeta'$ and $\zeta' \preceq \zeta$. Given a sequence of m-measurable partitions $(\zeta_n)_{n\in\mathbb{N}}$, there exists a m-measurable partition, denoted by

$\zeta = \bigvee_{n\in\mathbb{N}} \zeta_n$ and unique up to m-equivalence, such that $\zeta_n \preceq \zeta$ for every $n \in \mathbb{N}$, and if ζ' is a m-measurable partition such that $\zeta_n \preceq \zeta'$ for every $n \in \mathbb{N}$, then $\zeta \preceq \zeta'$.

Fix a measurable partition ζ of $(X, \mathscr{A}, m)$. The triple

$$\big(\,\zeta,\ \ \mathscr{A}_\zeta = \pi_\zeta(\widehat{\zeta}) = \{\pi_\zeta(A)\ :\ A \in \widehat{\zeta}\},\ \ m_\zeta = (\pi_\zeta)_* m : A \mapsto m(\pi_\zeta^{-1}(A))\,\big)$$

is a standard probability space, called the *factor space* of $(X, \mathscr{A}, m)$ by ζ. In particular, saying that for m_ζ-almost every $z \in \zeta$, the set z satisfies some property and for m-almost $x \in X$, the set $\zeta(x)$ satisfies this property are equivalent. For m_ζ-almost every $z \in \zeta$, there exists a unique standard probability measure $(\mathscr{A}_z, m_z)$ on z, such that

- for every $\mathscr{A}$-measurable map $f : X \to \mathbb{C}$, the map $f_{|z}$ is $\mathscr{A}_z$-measurable for m_ζ-almost every z in ζ, and $z \mapsto \int_z f_{|z}\, dm_z$ is m_ζ-measurable;
- the measure m disintegrates with respect to the projection π_ζ, with family of *conditional measures* on the fibers the family $(m_z)_{z\in\zeta}$: for every $\mathscr{A}$-measurable map $f : X \to \mathbb{C}$,

$$\int_X f\, dm = \int_{z\in\zeta} \int_z f_{|z}\, dm_z\, dm_\zeta(z)\ .$$

We only have to check these properties for f the characteristic functions of elements of $\mathscr{A}$, and the last integral is equal to $\int_{x\in X} \int_{\zeta(x)} f_{|\zeta(x)}\, dm_{\zeta(x)}\, dm(x)$. If ζ is finite or countable, then for every $z \in \zeta$ such that $m(z) \neq 0$, we have $m_z = \frac{1}{m(z)} m_{|z}$.

Fix a measure-preserving transformation $\varphi : X \to X$. We denote by $\varphi^{-1}\zeta$ the m-measurable partition of X by the preimages of the elements of ζ by φ. We say that ζ is *generating* under φ if the m-measurable partition $\bigvee_{k\in\mathbb{N}} \varphi^{-k}\zeta$ is m-equivalent to the partition by points. For all measurable partitions ζ and ζ' of X, define the *entropy of* ζ' *relative to* ζ as

$$H_m(\zeta' \,|\, \zeta) = \int_{x\in X} -\log m_{\zeta(x)}(\zeta'(x) \cap \zeta(x))\, dm(x)\ ,$$

with the convention that $-\log 0 = +\infty$. Note that if ζ and ζ' are finite or countable, then

$$H_m(\zeta' \,|\, \zeta) = \sum_{z\in\zeta,\, z'\in\zeta',\, m(z)\neq 0} -m(z' \cap z) \log \frac{m(z' \cap z)}{m(z)}\ ,$$

as more usual. If $\zeta \preceq \varphi^{-1}\zeta$, define the *entropy of* φ *relative to* ζ as

$$h_m(\varphi, \zeta) = H_m(\varphi^{-1}\zeta \,|\, \zeta)\ .$$

The *(metric) entropy* $h_m(\varphi)$ of φ with respect to m is the upper bound of $h_m(\varphi, \zeta)$ for ζ a measurable partition such that $\zeta \preceq \varphi^{-1}\zeta$. The *(metric) entropy* $h_m(\phi)$ of a flow $\phi = (\phi_t)_{t\in\mathbb{R}}$ of measure-preserving transformations of X is $h_m(\phi) = h_m(\phi_1)$. We have $h_m(\phi_t) = |t| h_m(\phi)$ for every $t \neq 0$.

6.2 Proof of the variational principle

The notation for this proof is the following one. Let $\delta = \delta_{\Gamma,\,F} \in\,]0, +\infty[$ and let $\widetilde{m}_F$ be a Gibbs measure on $T^1\widetilde{M}$ for Γ with potential F: there exist Patterson densities $(\mu^\iota_x)_{x\in\widetilde{M}}$ and

$(\mu_x)_{x\in\widetilde{M}}$ of dimension δ for $(\Gamma, F\circ\iota)$ and (Γ, F) such that, using the Hopf parametrisation $T^1\widetilde{M}\to\partial^2_\infty\widetilde{M}\times\mathbb{R}$ identifying v and (v_-, v_+, t),

$$d\,\widetilde{m}(v) = e^{C_{F\circ\iota-\delta,\,v_-}(x,\,\pi(v))+C_{F-\delta,\,v_+}(x,\,\pi(v))}d\mu^\iota_x(v_-)\,d\mu_x(v_+)\,dt \tag{79}$$

for any $x\in\widetilde{M}$. Let m_F be the induced measure on T^1M (see Subsection 3.7). Denote by $(\mu_{W^{su}(v)})_{v\in T^1\widetilde{M}}$ and $(\mu_{W^{su}(v)})_{v\in T^1M}$ the families of measures on the strong unstable leaves of $T^1\widetilde{M}$ and T^1M associated with the Patterson density $(\mu_x)_{x\in\widetilde{M}}$ as in Subsection 3.9.

Before beginning the proof of Theorem 6.1, we introduce another cocycle. For all v, w in the same strong unstable leaf in T^1M, define

$$c_F(v,w) = \lim_{t\to+\infty}\int_0^t \big(F(\phi_{-s}v)-F(\phi_{-s}w)\big)\;ds\;, \tag{80}$$

which exists since F is Hölder-continuous and $d(\phi_{-t}v,\phi_{-t}w)$ tends exponentially fast to 0. Note that c_F is unchanged by adding a constant to F, and that it satisfies the cocycle property

$$c_F(v,v')+c_F(v',v'')=c_F(v,v'')$$

for all v, v', v'' in the same strong unstable leaf in T^1M. For all $t\in\mathbb{R}$ and v, w in the same strong unstable leaf in T^1M, we have

$$c_F(\phi_tv,\phi_tw) = c_F(v,w)+\int_0^t\big(F(\phi_sv)-F(\phi_sw)\big)\;ds\;. \tag{81}$$

The relation between this cocycle and the Gibbs cocycle is the following. For all v, w in the same strong unstable leaf in T^1M, for all lifts $\widetilde{v}$ and $\widetilde{w}$ of v and w, respectively, in the same strong unstable leaf in $T^1\widetilde{M}$, we have, for every $\sigma\in\mathbb{R}$,

$$c_F(v,w) = -\,C_{F\circ\iota-\sigma,\,\widetilde{v}_-}(\pi(\widetilde{v}),\pi(\widetilde{w}))\;. \tag{82}$$

This follows from the definition of the Gibbs cocycle, since $x=\pi(\widetilde{v})$ and $y=\pi(\widetilde{w})$ being in the same horosphere centered at $\widetilde{v}_-$ implies that $C_{F\circ\iota-\sigma,\widetilde{v}_-}(x,y)=C_{F\circ\iota,\widetilde{v}_-}(x,y)$, and since the map $t\mapsto\phi_t(-\widetilde{v})=-\,\phi_{-t}(\widetilde{v})$ from $[0,+\infty[$ to $\widetilde{M}$ is the geodesic ray from x to $\widetilde{v}_-$.

Step 1. The first step of the proof of Theorem 6.1 is a construction of measurable partitions with nice geometric properties which allow entropy computations.

Here is what we mean by "nice geometric properties". Given a probability measure m on T^1M, a partition ζ of T^1M is called *m-subordinated* to the strong unstable partition $\mathscr{W}^{su}$ of T^1M if there exists a measurable fundamental domain $\mathscr{F}_\Gamma$ for the action of Γ on T^1M such that, for m-almost every v in T^1M, the set $\zeta(v)$ is a relatively compact neighbourhood of v in $W^{su}(v)$, and there exists a measurable subset $\widetilde{\zeta(v)}$ of $\mathscr{F}_\Gamma$, contained in one strong unstable leaf in $T^1\widetilde{M}$, such that the restriction of the canonical projection $T^1\widetilde{M}\to T^1M$ to $\widetilde{\zeta(v)}$ is a bijection onto $\zeta(v)$.

The next result constructs a measurable partition, subordinated to the strong unstable foliation and realizing the upper bound in the definition of the entropy, for every ergodic invariant probability measure on T^1M. It is due to [LeS] for Anosov transformations of compact manifolds, and the adaptation we will use to our geodesic flows is the content of the propositions 1 and 4 of [OtP].

Proposition 6.4 (Otal-Peigné) *Let m be a probability measure on T^1M invariant under the geodesic flow $(\phi_t)_{t\in\mathbb{R}}$, and let $\tau > 0$ be such that ϕ_τ is ergodic for m. Then there exists a m-measurable partition ζ of T^1M, which is generating under ϕ_τ and subordinated to $\mathscr{W}^{su}$, such that $(\phi_\tau^{-1}\zeta)(v)$ is contained in $\zeta(v)$ for m-almost every v, and such that*

$$h_m(\phi_\tau) = h_m(\phi_\tau, \zeta)\ .$$

Remark. The assumptions of [OtP] and ours are the same except that we allow Γ to have fixed points on $\widetilde{M}$, and our definition of a subordinated partitions is slightly stronger. Here are a few comments. The proof of Proposition 1 in [OtP] starts by fixing $u \in T^1M$ in the support of the measure m. For every $r > 0$, the authors consider the standard dynamical neighbourhood U_r of u defined by

$$U_r = \bigcup_{|t|<r} \phi_t\Big(\bigcup_{v\in B^{ss}(u,r)} B^{su}(v,r)\Big)\ ,$$

where $B^{ss}(u,r)$ is the open r-neighbourhood of u in its strong stable leaf for the induced Riemannian metric, and similarly for $B^{su}(v,r)$. Noting that U_r is the domain of a local chart of the foliation $\mathscr{W}^{su}$ if r is small enough, they define ζ as $\bigvee_{n=0}^{+\infty} \phi_{n\tau}\widehat{\zeta'}$ where ζ' is the partition of T^1M whose elements are $T^1M - U_r$ and the local leaves of $\mathscr{W}^{su}$ in U_r. They then prove that for Lebesgue-almost every $r > 0$ such that $4r$ is strictly less than the injectivity radius of M at the origin $\pi(u)$ of u, the partition ζ is as required.

In our case, starting with the same u, we first consider a lift $\widetilde{u}$ of u in $T^1\widetilde{M}$. The neighbourhood $\widetilde{U}_r$ of $\widetilde{u}$ constructed as above is then invariant under the stabiliser Γ_u of u in Γ. We take r small enough such that the images of $\widetilde{U}_{4r}$ by the elements of the finite set $\Gamma_{\pi(u)} - \Gamma_u$ are pairwise disjoint (where $\Gamma_{\pi(u)}$ is the stabiliser of $\pi(u)$ in Γ) and such that the restriction to $\widetilde{U}_{4r}$ of the canonical projection from $T^1\widetilde{M}$ to T^1M induces an homeomorphism from $\Gamma_u\backslash\widetilde{U}_{4r}$ onto an open neighbourhood of u. The construction in [OtP] then provides the above result, even with our slightly stronger definition of subordinated partitions.

Furthermore, the assertion in Proposition 6.4 that $(\phi_\tau^{-1}\zeta)(v)$ is contained in $\zeta(v)$ for m-almost every v is not stated in [OtP, Prop. 1] but is proven there. It implies that $\zeta \preceq \phi_{-\tau}\zeta$ (as needed for the computation of the relative entropy), but we will need an actual containment and not a containment up to a set of m-measure 0 in the third step of the proof.

Step 2. In order to apply Proposition 6.4 and the formula giving the entropy relative to a measurable partition, we need to be able to compute the conditional measures. The second step of the proof of Theorem 6.1 shows that a family of conditional measures $(m_z)_{z\in\zeta}$ of a finite Gibbs measure (normalised to be a probability measure) with respect to a subordinated measurable partition is a family of measures absolutely continuous with respect to (the restrictions to the elements of the partition of) the measures $\mu_{W^{su}(v)}$ on the strong unstable leaves, constructed using the above cocycle. The following lemma is a generalisation (up to appropriate normalisations) of Proposition 3.17, when the partition was by the whole strong unstable leaves.

Lemma 6.5 *Assume that m_F is finite and let $m^F = \frac{m_F}{\|m_F\|}$. Let ζ be a m^F-measurable partition of T^1M which is m^F-subordinated to $\mathscr{W}^{su}$. Then a family of conditional measures*

$(m^F_z)_{z\in\zeta}$ *of* m^F *with respect to* ζ *is given by*

$$dm^F_z(w) = \frac{\mathbb{1}_z(w)}{\int_z e^{-c_F(w,u)}\, d\mu_{W^{su}(v)}(u)}\, d\mu_{W^{su}(v)}(w)\ , \tag{83}$$

where $w\in z\in\zeta$, *for any* $v\in T^1M$ *such that* $z\subset W^{su}(v)$.

Note that when F is constant (and in particular when m_F is the Bowen-Margulis measure), m^F_z is just the normalised restriction to z of $\mu_{W^{su}(v)}$.

Proof. For every relatively compact Borel subset z of a strong unstable leaf $W^{su}(v)$ in T^1M, such that the support of $\mu_{W^{su}(v)}$ meets the interior of z, define a finite measure m'_z on z by

$$dm'_z(w) = \frac{\mathbb{1}_z(w)\ d\mu_{W^{su}(v)}(w)}{\int_z e^{-c_F(w,u)}\, d\mu_{W^{su}(v)}(u)}\ :$$

the denominator is positive and finite and the numerator is a finite measure. Furthermore, m'_z is a Borel probability measure on z, since by the cocycle property of c_F,

$$\|m'_z\| = \frac{\int_z e^{-c_F(v,w)}\, d\mu_{W^{su}(v)}(w)}{\int_z e^{-c_F(v,u)}\, d\mu_{W^{su}(v)}(u)} = 1\ .$$

Let us check the two properties for $(m'_z)_{z\in\zeta}$ to be a family of conditional measures for m^F with respect to ζ.

In order to check the measurability property, let $f: T^1M\rightarrow\mathbb{C}$ be a measurable map, that we may assume to be bounded with compact support. For every $n\in\mathbb{N}$, let X_n be the Borel subset of points of $T^1\widetilde M$ whose stabiliser in Γ has cardinality n, which is Γ-invariant and ϕ-invariant. Since m_F is finite, hence ergodic by Corollary 5.14, we fix $n_0\in\mathbb{N}$ such that X_{n_0} has full measure for $\widetilde m_F$. We may hence assume that f vanishes outside the image of X_{n_0} in T^1M. Up to multiplying by n_0 the maps and measures when lifting them to $T^1\widetilde M$ (see Subsection 2.6), we may assume that $n_0=1$. For m^F-almost every $v\in T^1M$, since $z=\zeta(v)$ is a measurable subset of T^1M, the restriction of f to z is measurable, and

$$v\mapsto\int_z f_{|z}\ dm'_z = \int_{w\in T^1M}\frac{\mathbb{1}_{\zeta(v)}(w)f(w)}{\int_{u\in T^1M}\mathbb{1}_{\zeta(v)}(u)\, e^{-c_F(w,u)}\, d\mu_{W^{su}(v)}(u)}\, d\mu_{W^{su}(v)}(w)$$

is clearly measurable, since $v\mapsto\mu_{W^{su}(v)}$ is weak star continuous and $\mathbb{1}_{\zeta(v)}(w)=\mathbb{1}_{\zeta(w)}(v)$.

In order to check the disintegration property, we give a sequence of equalities, whose reading guide (to be read simultaneously with the equalities below) is the following:

• for the first equality, we use the definition of the measures $(m'_z)_{z\in\zeta}$ and the topological properties of the elements of the partition ζ;

• for the second equality, we use a lifting by the canonical projection $p: T^1\widetilde M\rightarrow T^1M$; we denote by $\mathscr{F}_\Gamma$ and $\widetilde z$ the fundamental domain of Γ and lift of (almost every) $z\in\zeta$ given by the definition of a subordinated partition, and we define $\widetilde f=\mathbb{1}_{\mathscr{F}_\Gamma}f\circ p$; we also use Equation (82) and the fact that $\widetilde{\zeta(p(v))}$ is indeed contained in $W^{su}(v)$ for $\widetilde m_F$-almost every $v\in\mathscr{F}_\Gamma$ by the property of the partition ζ; note that unless the function which is integrated is 0, we have $w_-=v_-$;

• for the third equality, fixing $x\in\widetilde M$, we apply the definition of the measures $\mu_{W^{su}(v)}$ (see Equation (42)) and $\widetilde m_F$ (see Equation (79)); we use the Hopf parametrisation $v=(v_-,v_+,t)$ with respect to x and the homeomorphism $w\mapsto w_+$ from $W^{su}(v)$ to $\partial_\infty\widetilde M-\{v_-\}$;

- for the fourth equality, we use the cocycle equality

$$C_{F\circ\iota-\delta,\,v_-}(x,\pi(v)) = C_{F\circ\iota-\delta,\,v_-}(x,\pi(w)) + C_{F\circ\iota-\delta,\,v_-}(\pi(w),\pi(v))\ ;$$

we also use that since belonging to the same element of ζ is a symmetric relation, we have

$$\mathbb{1}_{\widetilde{\zeta(p(v))}}(w) = \mathbb{1}_{\zeta(p(v))}(p(w)) = \mathbb{1}_{\zeta(p(v))}(p(w)) = \mathbb{1}_{\widetilde{\zeta(p(w))}}(v)$$

for $\widetilde{m}_F$-almost every v; furthermore, unless the function which is integrated is 0, we have $w_- = v_-$ and $W^{su}(v) = W^{su}(w)$, and we denote $w = (v_-, w_+, t)$;

- for the fifth equality, we apply Fubini's theorem (exchanging the integrations on v_+ and w_+);
- for the last ones, again since the map $v \mapsto v_+$ from $W^{su}(w)$ to $\partial_\infty \widetilde{M} - \{w_-\}$ is a homeomorphism, we use the Hopf parametrisation $w = (w_- = v_-, w_+, t)$ with respect to x and the fact that the ratio of two integrals is 1:

$$\int_{v\in T^1M}\int_{w\in\zeta(v)} f(w)\, dm'_{\zeta(v)}(w) dm^F(v)$$

$$= \int_{v\in T^1M}\int_{w\in W^{su}(v)} \frac{\mathbb{1}_{\zeta(v)}(w) f(w)\ d\mu_{W^{su}(v)}(w)\ dm_F(v)}{\int_{u\in W^{su}(v)} \mathbb{1}_{\zeta(v)}(u)\, e^{-c_F(w,u)}\ d\mu_{W^{su}(v)}(u)\ \|m_F\|}$$

$$= \int_{v\in T^1\widetilde{M}}\int_{w\in W^{su}(v)} \frac{\mathbb{1}_{\widetilde{\zeta(p(v))}}(w) \widetilde{f}(w)\ d\mu_{W^{su}(v)}(w)\ d\widetilde{m}_F(v)}{\int_{u\in W^{su}(v)} \mathbb{1}_{\widetilde{\zeta(p(v))}}(u)\, e^{C_{F\circ\iota-\delta,\,w_-}(\pi(w),\,\pi(u))}\ d\mu_{W^{su}(v)}(u)\ \|m_F\|}$$

$$= \int_{\substack{t\,\in\,\mathbb{R}\\ v_-\,\in\,\partial_\infty\widetilde{M}\\ v_+\,\in\,\partial_\infty\widetilde{M}\\ v_+\,\neq\, v_-}}\int_{\substack{w_+\,\in\,\partial_\infty\widetilde{M}\\ w_+\,\neq\, v_-}} \frac{\mathbb{1}_{\widetilde{\zeta(p(v))}}(w)\widetilde{f}(w)\, e^{C_{F-\delta,\,w_+}(x,\,\pi(w))+C_{F\circ\iota-\delta,\,v_-}(x,\,\pi(v))+C_{F-\delta,\,v_+}(x,\,\pi(v))}}{\int_{u\in W^{su}(v)} \mathbb{1}_{\widetilde{\zeta(p(v))}}(u)\, e^{C_{F\circ\iota-\delta,\,w_-}(\pi(w),\,\pi(u))+C_{F-\delta,\,u_+}(x,\,\pi(u))}\ d\mu_x(u_+)}$$

$$\frac{d\mu_x(w_+) d\mu^\iota_x(v_-) d\mu_x(v_+) dt}{\|m_F\|}$$

$$= \int_{\substack{t\,\in\,\mathbb{R}\\ v_-\,\in\,\partial_\infty\widetilde{M}\\ v_+\,\in\,\partial_\infty\widetilde{M}\\ v_+\,\neq\, v_-}}\int_{\substack{w_+\,\in\,\partial_\infty\widetilde{M}\\ w_+\,\neq\, v_-}} \frac{\mathbb{1}_{\widetilde{\zeta(p(w))}}(v)\widetilde{f}(w)\, e^{C_{F\circ\iota-\delta,\,w_-}(\pi(w),\,\pi(v))+C_{F-\delta,\,v_+}(x,\,\pi(v))}}{\int_{u\in W^{su}(w)} \mathbb{1}_{\widetilde{\zeta(p(w))}}(u)\, e^{C_{F\circ\iota-\delta,\,w_-}(\pi(w),\,\pi(u))+C_{F-\delta,\,u_+}(x,\,\pi(u))}\ d\mu_x(u_+)}$$

$$\frac{e^{C_{F-\delta,\,w_+}(x,\,\pi(w))+C_{F\circ\iota-\delta,\,v_-}(x,\,\pi(w))}\ d\mu_x(w_+) d\mu^\iota_x(v_-) d\mu_x(v_+) dt}{\|m_F\|}$$

$$= \int_{\substack{t\,\in\,\mathbb{R}\\ v_-\,\in\,\partial_\infty\widetilde{M}\\ w_+\,\in\,\partial_\infty\widetilde{M}\\ w_+\,\neq\, v_-}} \widetilde{f}(w) \frac{\int_{\substack{v_+\,\in\,\partial_\infty\widetilde{M}\\ v_+\,\neq\, w_-}} \mathbb{1}_{\widetilde{\zeta(p(w))}}(v)\, e^{C_{F\circ\iota-\delta,\,w_-}(\pi(w),\,\pi(v))+C_{F-\delta,\,v_+}(x,\,\pi(v))}\ d\mu_x(v_+)}{\int_{u\in W^{su}(w)} \mathbb{1}_{\widetilde{\zeta(p(w))}}(u)\, e^{C_{F\circ\iota-\delta,\,w_-}(\pi(w),\,\pi(u))+C_{F-\delta,\,u_+}(x,\,\pi(u))}\ d\mu_x(u_+)}$$

$$\frac{e^{C_{F-\delta,\,w_+}(x,\,\pi(w))+C_{F\circ\iota-\delta,\,v_-}(x,\,\pi(w))}\ d\mu_x(w_+) d\mu^\iota_x(v_-) dt}{\|m_F\|}$$

$$= \int_{w\in T^1\widetilde{M}} \widetilde{f}(w)\, \frac{dm_F(w)}{\|m_F\|} = \int_{T^1M} f\, dm^F\ .$$

This is exactly the required disintegration property, and ends the proof of Lemma 6.5. $\square$

Step 3. In this penultimate step, we prove that for every ϕ-ergodic ϕ-invariant probability measure m on T^1M for which the negative part of F is integrable, its metric pressure $P_{\Gamma,F}(m)$ is at most equal to the critical exponent $\delta = \delta_{\Gamma,\,F}$, with equality if and only if the given Gibbs measure m_F is finite, in which case m is the normalisation of m_F to a probability measure. The interplay between these two measures m_F and m is crucial in the following result.

Lemma 6.6 *Let m be a ϕ-ergodic ϕ-invariant probability measure on T^1M such that $\int_{T^1M} F_-\,dm < +\infty$. Let $\tau > 0$ be such that ϕ_τ is ergodic with respect to m, and let ζ be a partition associated with (m,τ) by Proposition 6.4.*

(1) The real number

$$G(v) = -\log \int_{\zeta(v)} e^{-c_F(v,\,w)} d\mu_{W^{su}(v)}(w)$$

and the measure $m^F_{\zeta(v)}$ given by Equation (83) are defined for m-almost every v in T^1M, and we have, for m-almost every v in T^1M,

$$-\log m^F_{\zeta(v)}((\phi_\tau^{-1}\zeta)(v)) = \tau\delta + \int_0^{-\tau} F(\phi_t v)\;dt\;+G(\phi_\tau v) - G(v)\;. \tag{84}$$

(2) We have

$$\int_{T^1M} -\log m^F_{\zeta(v)}((\phi_\tau^{-1}\zeta)(v))\;dm(v) = \tau\delta - \tau\int_{T^1M} F\;dm\;. \tag{85}$$

(3) If $\int_{T^1M} F_-\;dm_F < +\infty$ and if m_F is finite, then, with $m^F = \frac{m_F}{\|m_F\|}$, we have

$$P_{\Gamma,F}(m^F) = \delta\;.$$

(4) We have $P_{\Gamma,F}(m) \le \delta$, hence

$$P(\Gamma,F) \le \delta\;.$$

(5) If $P_{\Gamma,F}(m) = \delta$, then m_F is finite and $m = m^F$, where $m^F = \frac{m_F}{\|m_F\|}$.

Proof. We will follow the same scheme of proof as the one for $F \equiv 0$ in [OtP].

(1) Note that the measure $\mu_{W^{su}(v)}$ is defined for every $v \in T^1M$. Since m is finite and ϕ-ergodic, m-almost every v belongs to the (topological) nonwandering set $\Omega\Gamma$ of the geodesic flow in T^1M. Recall that $\Omega\Gamma$ coincides with the set of elements $v \in T^1M$ whose lifts to $T^1\widetilde{M}$ have both endpoints in $\Lambda\Gamma$. Hence for m-almost every v, the element v belongs to the support of $\mu_{W^{su}(v)}$, by Equation (44). Furthermore, $\zeta(v)$ is a relatively compact neighbourhood of v in $W^{su}(v)$ for m-almost every v, since ζ is m-subordinated to the strong unstable foliation. Hence the measure

$$dm^F_{\zeta(v)}(w) = \frac{\mathbb{1}_{\zeta(v)}(w)}{\int_{\zeta(v)} e^{-c_F(w,\,u)}\;d\mu_{W^{su}(v)}(u)}\;d\mu_{W^{su}(v)}(w)$$

is well defined for m-almost every v. (Note that if m_F is finite and if ζ is also m^F-measurable and m^F-subordinated to $\mathscr{W}^{su}$, then $m^F_{\zeta(v)}$ is, by Lemma 6.5, the conditional

measure of m^F on $\zeta(v)$, but we are not assuming m_F to be finite in Assertion (1), (2) or (4).)

Recall that $(\phi_\tau^{-1}\zeta)(v)$ is contained in $\zeta(v)$ for m-almost every v by Proposition 6.4, and that by definition

$$(\phi_\tau^{-1}\zeta)(v) = \phi_{-\tau}(\zeta(\phi_\tau v)) \;.$$

By Equation (81), we have

$$c_F(\phi_{-\tau}w, v) = c_F(w, \phi_\tau v) + \int_0^{-\tau} \big(F(\phi_t w) - F(\phi_{t+\tau} v)\big)\, dt \;.$$

By Equation (45), we have

$$d\mu_{W^{su}(v)}(\phi_{-\tau}w) = e^{\int_0^\tau (F(\phi_{s-\tau}w)-\delta)\, dt}\, d\mu_{W^{su}(\phi_\tau v)}(w) = e^{\int_{-\tau}^0 (F(\phi_t w)-\delta)\, dt}\, d\mu_{W^{su}(\phi_\tau v)}(w) \;.$$

By the cocycle property, we have $c_F(\phi_{-\tau}w, u) = c_F(\phi_{-\tau}w, v) + c_F(v, u)$ and $c_F(w, \phi_\tau v) = -c_F(\phi_\tau v, w)$. Hence for m-almost every v, we have

$$\begin{aligned}
m^F_{\zeta(v)}((\phi_\tau^{-1})\zeta(v)) &= \int_{\phi_{-\tau}(\zeta(\phi_\tau(v)))} \frac{d\mu_{W^{su}(v)}(w)}{\int_{\zeta(v)} e^{-c_F(w,\,u)}\, d\mu_{W^{su}(v)}(u)} \\
&= \int_{\zeta(\phi_\tau(v))} \frac{e^{c_F(\phi_{-\tau}w,\,v)}\, d\mu_{W^{su}(v)}(\phi_{-\tau}w)}{\int_{\zeta(v)} e^{-c_F(v,u)}\, d\mu_{W^{su}(v)}(u)} \\
&= \frac{\int_{\zeta(\phi_\tau(v))} e^{c_F(w,\,\phi_\tau v) + \int_0^{-\tau}(F(\phi_t w)-F(\phi_t v))\, dt + \int_{-\tau}^0 (F(\phi_t w)-\delta)\, dt}\, d\mu_{W^{su}(\phi_\tau v)}(w)}{\int_{\zeta(v)} e^{-c_F(v,\,u)}\, d\mu_{W^{su}(v)}(u)} \\
&= e^{-\tau\delta - \int_0^{-\tau} F(\phi_t v)\, dt\, - G(\phi_\tau v) + G(v)} \;.
\end{aligned}$$

This proves Assertion (1).

(2) We are going to use the following quite classical result that we state without proof (see for instance [OtP, Lemme 8]).

Claim. *Let T be a measure-preserving transformation of a probability space $(X, \mathscr{A}, \mu)$, let $H : X \to \mathbb{R}$ be a measurable map such that $h = H \circ T - H$ has integrable negative part. Then h is integrable and $\int_X h\, dm = 0$.*

By Assertion (1), for m-almost every $v \in T^1M$, since $m^F_{\zeta(v)}$ is a probability measure, we have

$$G \circ \phi_\tau(v) - G(v) \geq -\tau\delta + \int_0^\tau F(\phi_{-t}v)\, dt \;.$$

Since the negative part of F is m-integrable, by Fubini's theorem and the ϕ-invariance of m, the map $g = G \circ \phi_\tau - G$ hence has m-integrable negative part. Applying the above claim, we may integrate Equation (84) over $v \in T^1M$ for the measure m. The vanishing of $\int_{T^1M} g\, dm$ then yields Equation (85).

(3) If m_F is finite, then its normalised measure m^F is an ergodic probability measure on T^1M (see Corollary 5.14). Hence there exists $\tau > 0$ such that ϕ_τ is ergodic for m^F (see Lemma 6.3). If $\int_{T^1M} F_-\, dm_F < +\infty$, applying Equation (85) to $m = m^F$ and $\zeta = \zeta'$ a m^F-measurable partition associated with (m^F, τ) by Proposition 6.4, we have, by the

definition of the entropy of ϕ_τ relative to ζ' and since $(\phi_\tau^{-1}\zeta')(v)$ is contained in $\zeta'(v)$ for m^F-almost every v,

$$h_{m^F}(\phi_\tau, \zeta') = \int_{v \in T^1 M} -\log m^F_{\zeta'(v)}((\phi_\tau^{-1}\zeta')(v))\; dm^F(v) = \tau\delta - \tau \int_{T^1 M} F\; dm^F \;.$$

By the definition of the entropy of ϕ ith respect to m^F and by Proposition 6.4, we have

$$h_{m^F}(\phi) = \frac{1}{\tau} h_{m^F}(\phi_\tau) = \frac{1}{\tau} h_{m^F}(\phi_\tau, \zeta') = \delta - \int_{T^1 M} F\; dm^F \;.$$

This gives $P_{\Gamma,\,F}(m^F) = \delta$.

(4) For m-almost every $v \in T^1 M$, by Assertion (1), define $\psi(v) \in [0, +\infty]$ as

$$\psi(v) = \frac{m^F_{\zeta(v)}\big((\phi_\tau^{-1}\zeta)(v)\big)}{m_{\zeta(v)}\big((\phi_\tau^{-1}\zeta)(v)\big)} \quad \text{if} \quad m_{\zeta(v)}\big((\phi_\tau^{-1}\zeta)(v)\big) > 0\,, \quad \text{and} \quad \psi(v) = +\infty \quad \text{otherwise}\,.$$

Claim. **(Otal-Peigné)** *The map ψ is m-mesurable, the maps ψ and $\log\psi$ are m-integrable, and $\int_X \psi\; dm \leq 1$.*

Proof. The proof when $F = 0$ in [OtP, Fait 9] extends immediately, by replacing μ_v^+ in that reference by $\mu_{W^{su}(v)}$, and by noting, to prove the measurability of ψ as indicated, that $v \mapsto \mu_{W^{su}(v)}$ is weak star continuous as already mentioned, hence $v \mapsto \mu_{W^{su}(v)}(A)$ is measurable for every Borel subset A of $T^1 M$. □

By the definition of the (metric) entropy and by Proposition 6.4, we have as above

$$\tau h_m(\phi) = h_m(\phi_\tau) = h_m(\phi_\tau, \zeta) = \int_{T^1 M} -\log m_{\zeta(v)}((\phi_\tau^{-1}\zeta)(v))\; dm(v) \;.$$

Hence by Equation (85),

$$\int_{T^1 M} \log\psi\; dm = -\tau\delta + \tau \int_{T^1 M} F\; dm + \tau h_m(\phi) \;.$$

By Jensen's inequality and by the above Claim, we have

$$\int_{T^1 M} \log\psi\; dm \leq \log\Big(\int_{T^1 M} \psi\; dm\Big) \leq 0 \;.$$

Hence

$$P_{\Gamma,\,F}(m) = h_m(\phi) + \int_{T^1 M} F\; dm \leq \delta \;,$$

as required.

The last claim of Assertion (4) follows by the definition of $P(\Gamma, F)$ (with the restriction to ergodic probability measures)since m is arbitrary and the existence of $\tau > 0$ such that ϕ_τ is ergodic for m is guaranteed by Lemma 6.3.

(5) The equality case in Jensen's inequality implies that $\psi(v) = 1$ for m-almost every $v \in T^1 M$. Hence the measure $m^F_{\zeta(v)}$ and the conditional measure $m_{\zeta(v)}$ coincide on the σ-algebra generated by the restriction to $\zeta(v)$ of $\phi_\tau^{-1}\zeta$ for m-almost every $v \in T^1 M$.

Similarly by replacing ϕ_τ by $\phi_{k\tau}$, these measures coincide on the σ-algebra generated by the restriction to $\zeta(v)$ of $\phi_\tau^{-k}\zeta$ for m-almost every $v \in T^1M$. Since ζ is generating under ϕ_τ (see Proposition 6.4), these measures are equal, for m-almost every $v \in T^1M$.

First assume that m_F is finite. In particular the normalised measure m^F is ergodic by Corollary 5.14. To prove that m and m^F coincide, let us show, using Hopf's argument, that $m(f) = m^F(f)$ for every $f : T^1M \to \mathbb{R}$ continuous with compact support. Let A_f be the set of $v \in T^1M$ such that the limit $\lim_{t\to+\infty} \frac{1}{t}\int_0^t f(\phi_s v)\, ds$ exists and is equal to $m^F(f)$. By Birkhoff's ergodic theorem, A_f has full measure with respect to m^F. By the uniform continuity of f, the set A_f is saturated by the stable foliation. By the quasi-product structure (see the comment after Equation (56)), for every $v \in T^1M$, the intersection $A_f \cap W^{su}(v)$ has full measure for $\mu_{W^{su}(v)}$. By the definition of $m^F_{\zeta(v)}$, this implies that the intersection $A_f \cap \zeta(v)$ has full measure with respect to $m^F_{\zeta(v)}$. By the above equality of measures, we have that $A_f \cap \zeta(v)$ has full measure with respect to $m_{\zeta(v)}$, for m-almost every v. By definition of the conditional measures of m, this implies that A_f has full measure with respect to m. By Birkhoff's ergodic theorem applied this time to m, we have $m(f) = m^F(f)$, as required.

Let us prove that m_F is indeed finite. By the Hopf-Tsuji-Sullivan-Roblin theorems 5.3 and 5.4, the dynamical system $(T^1M, (\phi_t)_{t\in\mathbb{R}}, m_F)$ is either completely dissipative or completely conservative.

Assume that the first case holds. Let $f : T^1M \to \mathbb{R}$ be continuous with compact support. Then for m_F-almost every $v \in T^1M$, we have $\lim_{t\to+\infty} f(\phi_s v)\, ds = 0$. Considering the set

$$A'_f = \{v \in T^1M \ :\ \lim_{t\to+\infty} \frac{1}{t}\int_0^t f(\phi_s v)\, ds = 0\}\ ,$$

which has full measure with respect to m_F, we have as above that $m(f) = 0$ for every such $f : T^1M \to \mathbb{R}$. This contradicts the fact that m is a probability measure.

Assume that the second case holds. Then there exists a positive uniformly continuous map $\rho : T^1M \to \mathbb{R}$ with $\|\rho\|_{\mathbb{L}^1(m_F)} = 1$ such that, for m_F-almost every $v \in T^1M$, we have

$$\int_0^\infty \rho\circ\phi_{-t}(v)dt = \int_0^\infty \rho\circ\phi_t(v)dt = +\infty$$

(see for instance the proof of Assertion (h) of Proposition 5.5). The same argument as above, this time using Hopf's ratio ergodic theorem, gives that $\frac{m(f)}{m(\rho)} = \frac{m_F(f)}{m_F(\rho)}$. This implies that m_F is finite, since m is a probability measure. □

Step 4. In this last step of the proof of Theorem 6.1, we assume that F is bounded from below on $\Omega\Gamma$ and we prove the converse inequality $\delta \le P(\Gamma, F)$. Assuming this, Assertion (4) of Lemma 6.6 implies that $\delta = P(\Gamma, F)$. If m_F is finite and $\int_{T^1M} F_-\, dm_F < +\infty$, then the assertions (3) and (5) of Lemma 6.6 imply that the normalised Gibbs measure m^F is the unique equilibrium state. Otherwise, Assertion (5) of Lemma 6.6 also implies that there is no equilibrium state (since $m = m^F$ implies that the negative part of the potential F is integrable with respect to m^F if it is with respect to m). Therefore Theorem 6.1 follows from the following result.

Lemma 6.7 *If F is bounded from below on $\Omega\Gamma$, then $\delta_{\Gamma,F} \le P(\Gamma, F)$.*

Proof. We start the proof by recalling an alternative definition of the topological pressure of a flow on a compact metric space.

Let X be a compact metric space whose distance is denoted by d, let $\varphi = (\varphi_t)_{t\in\mathbb{R}}$ be a continuous flow of homeomorphisms of X, and let $G : X \to \mathbb{R}$ be a Hölder-continuous map.

For all $T \geq 0$ and $\epsilon > 0$, a subset E of X is (ϵ, T)-*separated* if for all distinct points x and y in E, there exists $t \in [0, T]$ such that $d(\varphi_t x, \varphi_t y) \geq \epsilon$. Define $P_{d,\,\epsilon,\,T,\,G}$ as the upper bound of

$$\sum_{x\in E} e^{\int_0^T G(\varphi_t x)\,dt}$$

on all (ϵ, T)-separated subsets E of X, and the *combinatorial pressure* of (G, φ) as

$$P_X(G,\varphi) = \sup_{\epsilon>0}\ \limsup_{T\to+\infty} \frac{1}{T}\ \log P_{d,\,\epsilon,\,T,\,G}\ .$$

The following variational principle says that the combinatorial pressure and the topological pressure coincide. We denote by $\mathscr{M}(X, \varphi)$ the weak star compact convex set of probability measures on X invariant by the flow φ.

Theorem 6.8 *(see for instance [Wal, Th 9.10]) We have*

$$P_X(G,\varphi) = \sup_{m\in\mathscr{M}(X,\varphi)} \Big(h_m(\varphi) + \int_X G\,dm\Big)\ .$$

Note that this result is stated for transformations in [Wal, Th 9.10], but its proof extends to flows.

Let us resume the proof of Lemma 6.7. We may assume that $\delta = \delta_{\Gamma,\,F} > 0$. We are going to construct, using Subsection 4.4, for every $\delta'' \in\,]0, \delta[$, a compact subset K of T^1M invariant by the geodesic flow, such that the combinatorial pressure of $(F_{|K}, \phi_{|K})$ on K satisfies

$$P_K(F_{|K}, \phi_{|K}) \geq \delta''\ .$$

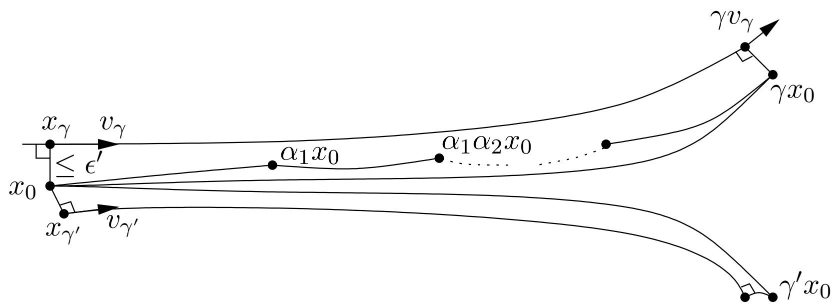
(We endow K with the distance d which is the restriction to K of the quotient of the Riemannian distance of $T^1\widetilde{M}$ (for the Sasaki metric).) Since $\mathscr{M}(K, \varphi)$ is contained in $\mathscr{M}_F(T^1M)$ (by considering measures on K as measures on T^1M with support in K), this implies that $P(\Gamma, F) \geq \delta_{\Gamma,\,F}$ as required in Lemma 6.7.

Let us now define the constants and objects we will use in the proof of Lemma 6.7. Let $\delta' \in\,]\delta'', \delta[$. Let $L \geq 5$ (which will be chosen large enough) and $\epsilon' \in\,]0, \frac{1}{12}[$ (which will be chosen small enough). Let $\epsilon \in\,]0, \frac{1}{16}]$ (depending only on ϵ' and tending to 0 as $\epsilon' \to 0$) be given by Lemma 2.7 with $\ell = 3$. Let $\theta > 0$ (depending only on ϵ) be defined by Proposition 4.6. Let $C = 6$. Let $x_0 \in \widetilde{M}$, $v_0 \in T^1_{x_0}\widetilde{M}$, $N > L$ and S a finite subset of Γ (depending on δ', θ, C, L) be given by Proposition 4.7. Let G be the subsemigroup of Γ generated by S.

By Proposition 4.6 (whose assumptions are satisfied by the assertions of Proposition 4.7), the semigroup G is free on S and for every $\gamma \in G$ of nonzero length ℓ as a word in the elements of S, we have

$$d(\gamma x_0, [x_0, \gamma^2 x_0]) \leq \epsilon \quad\text{and}\quad d(x_0, \gamma x_0) \geq (N - 2\epsilon)\ell \geq 3\ .$$

Hence by Lemma 2.7, every nontrivial element γ of G is hyperbolic and if x_γ is the closest point to x_0 on the translation axis Axe_γ of γ, then $d(x_0, x_\gamma) \le \epsilon'$ (see the picture above Lemma 2.7). Let v_γ be the unit tangent vector at x_γ pointing towards γx_γ, which is tangent to the translation axis Axe_γ (see the picture below).



Let $\mathrm{pr} : T^1\widetilde{M} \to T^1M = \Gamma\backslash T^1\widetilde{M}$ be the canonical projection. For every $\eta > 0$, denote by $\overline{\mathscr{N}}_\eta(A)$ the closed η-neighbourhood of a subset A of $\widetilde{M}$. Consider the Γ-invariant subset $\widetilde{K}$ of $\widetilde{M}$ defined by

$$\widetilde{K} = \Big\{ v \in T^1\widetilde{M} \;:\; \forall\, t \in \mathbb{R},\;\; \pi(\phi_t v) \in \Gamma \bigcup_{\alpha \in S} \overline{\mathscr{N}}_{\epsilon+\epsilon'}([x_0, \alpha x_0]) \Big\}\,.$$

Its image $K = \mathrm{pr}(\widetilde{K})$, which consists in the elements of T^1M whose orbit under the geodesic flow stays at distance at most $\epsilon + \epsilon'$ of $\bigcup_{\alpha\in S} \mathrm{pr}([x_0, \alpha x_0])$, is compact, since S is finite. Note that K (which depends on L and ϵ') is invariant under the geodesic flow, by construction.

Now, to finish the proof, for a fixed $\epsilon'' > 0$ small enough, we are going to construct, for every $k \in \mathbb{N}$, a $(\epsilon'', k(N+1))$-separated subset of K with a good control of the integral of the potential on its flow lines.

For every $k \ge 1$, let S_k be the set of words of length k in the elements of S. Let $\alpha_1, \dots, \alpha_k \in S$ and define $\gamma = \alpha_1 \dots \alpha_k \in S_k$. Note that $[x_0, \gamma x_0]$ is contained in

$$\overline{\mathscr{N}}_\epsilon([x_0, \alpha_1 x_0] \cup [\alpha_1 x_0, \alpha_1\alpha_2 x_0] \cup \dots \cup [\alpha_1 \dots \alpha_{k-1} x_0, \gamma x_0])$$

by Proposition 4.6.

We hence have, for every $t \in \mathbb{R}$,

$$\pi(\phi_t v_\gamma) \in Axe_\gamma = \gamma^{\mathbb{Z}}[x_\gamma, \gamma x_\gamma] \subset \Gamma \overline{\mathscr{N}}_{\epsilon'}([x_0, \gamma x_0]) \subset \Gamma \bigcup_{\alpha\in S} \overline{\mathscr{N}}_{\epsilon+\epsilon'}([x_0, \alpha x_0])\,.$$

This proves that v_γ belongs to $\widetilde{K}$ for every nontrivial γ in G.

Furthermore, if ϵ' (hence ϵ) is small enough, by Lemma 3.2, there exists $c \ge 0$ (depending only on ϵ, on the Hölder constants of $\widetilde{F}$ and on $\max_{\pi^{-1}(B(x_0,1))} |\widetilde{F}|$) such that we have, for all $k \in \mathbb{N}$ and $\alpha_1, \dots, \alpha_k \in S$,

$$\Big| \int_{x_0}^{\alpha_1 \dots \alpha_k x_0} \widetilde{F} \;- \sum_{i=1}^{k-1} \int_{x_0}^{\alpha_i x_0} \widetilde{F} \,\Big| \le c$$

and, for every $\gamma \in G$,

$$\Big|\int_{x_\gamma}^{\gamma x_\gamma} \widetilde{F} - \int_{x_0}^{\gamma x_0} \widetilde{F}\,\Big| \le c\,.$$

Hence, by Assertion (4) of Proposition 4.7, we have

$$\sum_{\gamma \in S_k} e^{\int_{x_\gamma}^{\gamma x_\gamma} \widetilde{F}} \ge e^{-2c}\Big(\sum_{\alpha \in S} e^{\int_{x_0}^{\alpha x_0} \widetilde{F}}\Big)^k \ge e^{\delta' k N - 2c}\,. \tag{86}$$

Let $C = \inf_{\pi^{-1}(\overline{\mathscr{N}}_{\epsilon+2\epsilon'}(\pi(\widetilde{\Omega}\Gamma)))} \widetilde{F}$, which is a finite constant since F is Hölder-continuous and bounded from below on $\Omega\Gamma$. With $\ell(\gamma) = d(x_\gamma, \gamma x_\gamma)$ the translation length of γ, by Proposition 4.6, we have

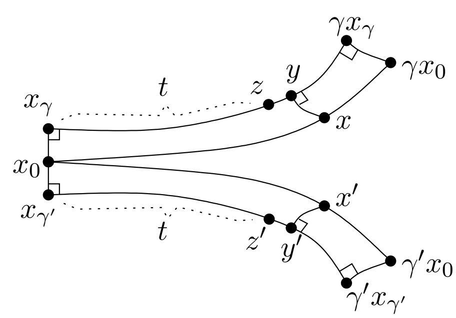
$$k(N - 2\epsilon) - 2\epsilon' \le d(x_0, \gamma x_0) - 2\epsilon' \le \ell(\gamma) \le d(x_0, \gamma x_0) \le k(N+1)\,.$$

Since $d(x_0, \pi(\widetilde{\Omega}\Gamma)) \le \epsilon'$, we have $\widetilde{K} \subset \pi^{-1}(\overline{\mathscr{N}}_{\epsilon+2\epsilon'}(\pi(\widetilde{\Omega}\Gamma)))$. Since $v_\gamma \in \widetilde{K}$ and $\widetilde{K}$ is invariant under the geodesic flow, we hence have $\int_{\ell(\gamma)}^{k(N+1)} \widetilde{F}(\phi_t v_\gamma)\,dt \ge C(k(1+2\epsilon)+2\epsilon')$. Therefore by Equation (86)

$$\sum_{\gamma \in S_k} e^{\int_0^{k(N+1)} \widetilde{F}(\phi_t v_\gamma)\,dt} = \sum_{\gamma \in S_k} e^{\int_{x_\gamma}^{\gamma x_\gamma} \widetilde{F} + \int_{\ell(\gamma)}^{k(N+1)} \widetilde{F}(\phi_t v_\gamma)\,dt} \ge e^{\delta' k(N + \frac{C(1+2\epsilon)}{\delta'}) - 2c + 2\epsilon' C}\,. \tag{87}$$

Lemma 6.9 *For every $\epsilon'' \in\,]0, \frac{1}{2} - 6\epsilon'[\,$, for all $k \in \mathbb{N}$ and $\gamma \neq \gamma'$ in S_k, there exists $t \in [0, k(N+1)]$ such that $d(\pi(\phi_t v_\gamma), \pi(\phi_t v_{\gamma'})) \ge \epsilon''$.*

Proof. Since $\epsilon \le \frac{1}{16}$, by the last claim of Proposition 4.6, there exists $x \in [x_0, \gamma x_0]$ and $x' \in [x_0, \gamma' x_0]$ such that $d(x_0, x) = d(x_0, x')$ and $d(x, x') \ge \frac{1}{2}$. Let y and y' be the closest points to x and x' on $[x_\gamma, \gamma x_\gamma]$ and $[x_{\gamma'}, \gamma' x_{\gamma'}]$ respectively, which satisfy $d(x, y), d(x', y') \le \epsilon'$ by convexity. Let $t = d(x_0, x) - 2\epsilon'$ and $z = \pi(\phi_t v_\gamma)$ and $z' = \pi(\phi_t v_{\gamma'})$.



By the triangle inequality, and since closest point maps do not increase the distances, we have

$$d(x_\gamma, z) = t = d(x_0, x) - 2\epsilon' \le d(x_\gamma, y) \le d(x_0, x)\,.$$

Hence $t \le d(x_\gamma, \gamma x_\gamma) = \ell(\gamma) \le k(N+1)$, and by the triangle inequality

$$\begin{aligned} d(z, x) &\le d(z, y) + d(y, x) = (d(x_\gamma, y) - d(x_\gamma, z)) + d(y, x) \\ &\le d(x_0, x) - (d(x_0, x) - 2\epsilon') + \epsilon' = 3\epsilon'\,. \end{aligned}$$

Similarly, $d(z', x') \le 3\epsilon'$, hence by the triangle inequality

$$d(z, z') \ge d(x, x') - d(x, z) - d(x', z') \ge \frac{1}{2} - 6\epsilon' \ge \epsilon''\,.\Box$$

Since $d_S(\phi_t v_\gamma, \phi_t v_{\gamma'}) \ge d(\pi(\phi_t v_\gamma), \pi(\phi_t v_{\gamma'}))$ by Equation (4), this lemma implies that the set $\widetilde{E}_k = \{v_\gamma \,:\, \gamma \in S_k\}$ is $(\epsilon'', k(N+1))$-separated.

Since K is compact and since the isometric action of Γ on $\widetilde{M}$ is proper, there exists $\epsilon'' \in]0, \frac{1}{2} - 6\epsilon'[$ (this interval is nonempty since $\epsilon' < \frac{1}{12}$) and $N_0 > 0$ (when Γ is torsion-free, we may take $N_0 = 1$ and $\epsilon'' > 0$ less than the lower bound on $\pi(K)$ of the injectivity radius of M) such that $\widetilde{E}_k$ may be subdivided in at most N_0 subsets that the map $\mathrm{pr} : T^1\widetilde{M} \to T^1 M$ inject into $(\epsilon'', k(N+1))$-separated subsets. For at least one of these sets, let us call it E_k, we have

$$\sum_{v \in E_k} e^{\int_0^{k(N+1)} F(\phi_t v)\, dt} \geq \frac{1}{N_0} \sum_{\gamma \in S_k} e^{\int_0^{k(N+1)} \widetilde{F}(\phi_t v_\gamma)\, dt} \geq \frac{e^{-2c+2\epsilon' C}}{N_0}\; e^{\delta' k(N+\frac{C(1+2\epsilon)}{\delta'})}\ .$$

Taking the logarithm, dividing by $k(N+1)$ and letting $k \to +\infty$, we have

$$P_K(F_{|K}, \phi_{|K}) \geq \delta' \frac{N + \frac{C(1+2\epsilon)}{\delta'}}{N+1}\ .$$

Since C is independent of K, taking L (hence $N > L$) large enough, we have $P_K(F_{|K}, \phi_{|K}) \geq \delta''$. This is what we wanted to prove. $\square$

Remark. As said previously, the assumption that $\widetilde{F}$ is bounded from below on $\widetilde{\Omega}\Gamma$ is not optimal in Lemma 6.7. In the above proof, replace the constant C by $C_N = \min_{\pi^{-1}(B(x_0,\, N+1+\epsilon+\epsilon'))} \widetilde{F}$ and note that $\widetilde{K}$ is contained in $\Gamma\pi^{-1}(B(x_0, N+1+\epsilon+\epsilon'))$ by construction. Then Equation (87) where C is replaced by C_N is still valid. If $\frac{C_N}{N}$ tends to 0 as $N \to +\infty$, then the conclusion of Lemma 6.7 still holds. But it is still not clear whether this weaker assumption is optimal or not.

7 The Liouville measure as a Gibbs measure

Let $(\widetilde{M}, \Gamma)$ be as in the beginning of Chapter 2: $\widetilde{M}$ is a complete simply connected Riemannian manifold, with dimension at least 2 and pinched sectional curvature at most -1 and Γ is a nonelementary discrete group of isometries of $\widetilde{M}$.

The aim of this chapter is to investigate when the Liouville measure of the geodesic flow on $T^1M = \Gamma\backslash T^1\widetilde{M}$ coincides (up to a multiplicative constant) with a Gibbs measure associated to some Hölder potential. This result is known when M is compact, but we provide a complete short proof in Subsection 7.2. Finding a necessary and sufficient condition in the noncompact case is still open. The approach that we follow in Subsection 7.3 actually allows us to get more than what was previously known on the subject. In particular, we obtain that if M is a Riemannian cover of a compact manifold, then the Liouville measure on T^1M is a Gibbs measure as soon as it sufficiently conservative (see below), but not necessarily finite.

Recall that the *Liouville measure* $\mathrm{vol}_{T^1\widetilde{M}}$ on $T^1\widetilde{M}$ is the Riemannian volume of the Sasaki metric on $T^1\widetilde{M}$ (see Subsection 2.3 for a definition). Equivalently, it disintegrates with respect to the projection $\pi : T^1\widetilde{M} \to \widetilde{M}$, with measure on the basis $\widetilde{M}$ the Riemannian volume $\mathrm{vol}_{\widetilde{M}}$ of $\widetilde{M}$ and conditional measures on the fibers $T^1_x\widetilde{M}$ the Riemannian spherical measure $\mathrm{vol}_{T^1_x\widetilde{M}}$ of the unit sphere of the Euclidean space $T_x\widetilde{M}$: for every $f \in \mathscr{C}_c(T^1\widetilde{M};\mathbb{R})$, we have

$$\int_{v\in T^1\widetilde{M}} f(v)\; d\,\mathrm{vol}_{T^1\widetilde{M}}(v) = \int_{x\in\widetilde{M}} \int_{v\in T^1_x\widetilde{M}} f(v)\; d\,\mathrm{vol}_{T^1_x\widetilde{M}}(v)\; d\,\mathrm{vol}_{\widetilde{M}}(x)\ .$$

It is invariant under the action of the geodesic flow (and the action of $\mathbb{Z}/2\mathbb{Z}$ by time reversal). It induces (see Subsection 2.6) a measure $\operatorname{vol}_{T^1M}$ on the Riemannian orbifold $T^1M = \Gamma\backslash T^1\widetilde{M}$, called the *Liouville measure* on T^1M, invariant under the geodesic flow (and time reversal), which disintegrates similarly over the Riemannian orbifold $M = \Gamma\backslash\widetilde{M}$ with fibers the Riemannian orbifolds $T^1_xM = \Gamma_{\widetilde{x}}\backslash T^1_{\widetilde{x}}\widetilde{M}$ where $\widetilde{x}$ is a lift of x, whose stabiliser in Γ is denoted by $\Gamma_{\widetilde{x}}$:

$$d\operatorname{vol}_{T^1M}(v) = \int_{x\in M} d\operatorname{vol}_{T^1_xM}(v)\; d\operatorname{vol}_M(x)\;.$$

In particular, the Liouville measure $\operatorname{vol}_{T^1M}$ is finite if and only if M has finite volume: since the fixed point set of a Riemannian isometry different from id of a connected Riemannian manifold has Riemannian measure 0, if M has dimension m, then

$$\operatorname{Vol}(T^1M) = \operatorname{Vol}(\mathbb{S}^{m-1})\;\operatorname{Vol}(M)\;.$$

It can happen that $\operatorname{vol}_{T^1M}$ is infinite and ergodic, see for example [Ree], where she proves that the Liouville measure on a Galois cover with covering group $\mathbb{Z}^d$ of a compact hyperbolic manifold is ergodic with respect to the geodesic flow if and only if $d \leq 2$.

Let us now define the potential on $T^1\widetilde{M}$ with respect to which the Liouville measure is a Gibbs measure in all known cases. Let $v \in T^1\widetilde{M}$ and $t \in \mathbb{R}$. Denote by $E^{su}(v)$ (respectively $E^{ss}(v)$) the tangent space $T_vW^{su}(v)$ at v to $W^{su}(v)$ (respectively $T_vW^{ss}(v)$ at v to $W^{ss}(v)$) and let $E^0(v) = \mathbb{R}\frac{d}{dt}_{|t=0}(\phi_t v)$ be the tangent space at v of its geodesic flow line. We have

$$T_vT^1\widetilde{M} = E^{su}(v)\;\oplus\;E^0(v)\;\oplus\;E^{ss}(v)\;.$$

Also define

$$E^u(v) = E^{su}(v)\;\oplus\;E^0(v)\quad\text{and}\quad E^s(v) = E^0(v)\;\oplus\;E^{ss}(v)\;.$$

For every $v \in T^1M$, we again denote by $E^{su}(v)$, $E^0(v)$, $E^{ss}(v)$ the images of $E^{su}(\widetilde{v})$, $E^0(\widetilde{v})$, $E^{ss}(\widetilde{v})$, respectively, by the canonical map $TT^1\widetilde{M} \to TT^1M = \Gamma\backslash TT^1\widetilde{M}$, where $\widetilde{v}$ is any lift of v by the canonical projection $T^1\widetilde{M} \to T^1M$.

For every $t \in \mathbb{R}$, the map ϕ_t on $T^1\widetilde{M}$ sends $W^{su}(v)$ to $W^{su}(\phi_t v)$. Choose two orthonormal bases (for the scalar product induced by the Sasaki metric) on $E^{su}(v)$ and $E^{su}(\phi_t v)$. We can therefore define the Jacobian of the restriction of ϕ_t to the smooth submanifold $W^{su}(v)$ of $T^1\widetilde{M}$ at the point v (that is, the absolute value of the determinant of the tangent map at v of $\phi_{t|W^{su}(v)}$, which does not depend on the choices of the orthonormal bases), and we denote it by $\widetilde{J}^{su}(v,t) > 0$. It is smooth in t and continuous in (v,t) (and studying its regularity in v will be one of the main points below). The flow property of the geodesic flow and the invariance of the strong unstable foliation under the geodesic flow give the following cocycle relation: for all $s,t \in \mathbb{R}$ and $v \in T^1\widetilde{M}$,

$$\widetilde{J}^{su}(v,0) = 1 \quad\text{and}\quad \widetilde{J}^{su}(v,s+t) = \widetilde{J}^{su}(\phi_t v,s)\;\widetilde{J}^{su}(v,t)\;. \tag{88}$$

Now, define the following map

$$\widetilde{F}^{su}(v) = -\frac{d}{dt}_{|t=0}\log\widetilde{J}^{su}(v,t) = \frac{d}{dt}_{|t=0}\log\widetilde{J}^{su}(\phi_t v,-t)\,. \tag{89}$$

The maps $v \mapsto \widetilde{J}^{su}(v,t)$ and $v \mapsto \widetilde{F}^{su}(v)$, being Γ-invariant, define maps $v \mapsto J^{su}(v,t)$ and $v \mapsto F^{su}(v)$ on T^1M (which, when Γ is torsion free, may be defined directly on T^1M). We have $\widetilde{F}^{su} \leq 0$ by the dilation properties of the geodesic flow on the unstable foliation. An estimation of the gap map $D_{F^{su}} : \widetilde{M} \times \partial^2_\infty \widetilde{M} \to \mathbb{R}$ (see Subsection 3.4) of the potential F^{su} would be interesting.

Note that $\widetilde{J}^{su}(v,t) = e^{(n-1)t}$ for all $v \in T^1\widetilde{M}$ and $t \in \mathbb{R}$ if M has dimension n and constant sectional curvature -1, so that $\widetilde{F}^{su}$ is then the constant map with value $-(n-1)$.

We will prove in this chapter the following results.

Theorem 7.1 *Let $\widetilde{M}$ be a complete simply connected Riemannian manifold, with dimension at least 2 and pinched negative sectional curvature. Assume that the derivatives of the sectional curvature are uniformly bounded. Then $\widetilde{F}^{su}$ is Hölder-continuous, bounded and reversible.*

Note that the new hypothesis of having bounded derivatives of sectional curvatures is satisfied if there exists a compact set in $\widetilde{M}$ whose images by the group $\operatorname{Isom}(\widetilde{M})$ cover $\widetilde{M}$, for instance if $\widetilde{M}$ has a cocompact discrete isometry group. In particular, the critical exponent $\delta_{\Gamma,\,F^{su}}$ is then finite (see Lemma 3.3 (iv)). We will then denote by $\widetilde{m}_{F^{su}}$ and $m_{F^{su}}$ the Gibbs measures on $T^1\widetilde{M}$ and T^1M associated with a fixed pair of Patterson densities of dimension $\delta_{\Gamma,\,F^{su}}$ for $(\Gamma, F^{su} \circ \iota)$ and (Γ, F^{su}).

Recall (see for instance [PoY, Chap. 16], [AaN]) that a measure preserving flow $(\psi_t)_{t\in\mathbb{R}}$ of a measured space (X,μ) is 2-*recurrent* if for every measurable subset B in X with $\mu(B) > 0$, there exists $n \in \mathbb{N} - \{0\}$ such that

$$\mu(\psi_{-n}B \cap B \cap \psi_n B) > 0\,.$$

For every $d \in \mathbb{N} - \{0\}$, the notion of d-recurrence (replacing the above centered formula by $\mu(B \cap \psi_{-n}B \cap \dots \cap \psi_{-dn}B) > 0$) has been introduced by Furstenberg in his proof of Szemeredi's theorem, and the 1-recurrence property is the complete conservativity property (see Subsection 5.2). Furstenberg proved that if μ is a probability measure, then $(\psi_t)_{t\in\mathbb{R}}$ is d-recurrent for every $d \in \mathbb{N} - \{0\}$. But when μ is infinite, there are examples of transformations which are 1-recurrent but not 2-recurrent (see for instance [AaN]).

Theorem 7.2 *Let $\widetilde{M}$ be a complete simply connected Riemannian manifold, with dimension at least 2 and pinched negative sectional curvature. Assume that the derivatives of the sectional curvature are uniformly bounded.*

If the geodesic flow of M is 2-recurrent with respect to the Liouville measure and if $\delta_{\Gamma,\,F^{su}} \leq 0$, then the Liouville measure is proportional to the Gibbs measure $m_{F^{su}}$ of the potential F^{su}. Furthermore, (Γ, F^{su}) is of divergence type and

$$P(\Gamma, F^{su}) = \delta_{\Gamma,\,F^{su}} = P_{Gur}(\Gamma, F^{su}) = 0\,.$$

The results of the article of Bowen-Ruelle [BoR] for Axiom A flows imply, in the case of the geodesic flow on the unit tangent bundle of a compact negatively curved Riemannian manifold, that its the Liouville measure coincides with a multiple of its Gibbs measure $m_{F^{su}}$. See in particular Remark 5.2 in loc. cit., and Subsection 7.2.

Remarks. (1) The inequality $\delta_{\Gamma,\,F^{su}} \leq 0$ is satisfied when M is compact, by Ruelle's inequality (see Equation (93) below, using the fact that $\widetilde{F}^{su}$ is bounded, hence $\delta_{\Gamma,\,F^{su}} =$

$P(\Gamma, F^{su})$ by the Variational principle theorem 6.1 (2)). Hence if Γ is a subgroup of a discrete cocompact group of isometries of $\widetilde{M}$, then $\delta_{\Gamma,\, F^{su}} \leq 0$ by Lemma 3.3 (iii).

It would be interesting to know whether this inequality remains true for any nonelementary discrete group of isometries Γ of $\widetilde{M}$.

(2) The assumptions that the derivatives of the sectional curvature are uniformly bounded and that $\delta_{\Gamma,\, F^{su}} \leq 0$ are satisfied if M is a Riemannian cover of a compact manifold, by the previous remark. Hence Theorem 1.2 in the introduction follows from Theorem 7.2.

(3) Rees [Ree] proved that the geodesic flow on a regular Riemannian cover of a hyperbolic manifold with covering group $\mathbb{Z}^d$ is ergodic (and hence completely conservative by [Aar, Prop. 1.2.1]) with respect to the Liouville measure if and only if $d \in \{0, 1, 2\}$. When $d = 1$, Aaronson (private communication) suspects that the geodesic flow is furthermore 2-recurrent.

It would be interesting to know when exactly the geodesic flow of a (non necessarily regular) Riemannian cover N' of a compact connected negatively curved Riemannian manifold N is ergodic or 2-recurrent with respect to the Liouville measure. For instance, if $N' = G' \backslash \widetilde{N}$ and $N = G \backslash \widetilde{N}$ with $\widetilde{N} \to N$ a universal Riemannian cover and G' a (non necessarily normal) subgroup of G, if S is a finite generating set of G and $X(G, S)$ is the Cayley graph of G with respect to S, is the geodesic flow of N' ergodic with respect to the Liouville measure if and only if the simple random walk (or an adequate one) on the Schreier graph $G' \backslash X(G, S)$ is recurrent ?

7.1 The Hölder-continuity of the (un)stable Jacobian

We will prove in this subsection the following Hölder regularity result stated in Theorem 7.3. As we make no compactness assumption, the choice of distances is important.

As explained in Subsection 2.4, we endow $T\widetilde{M}$ and then $TT\widetilde{M}$ with their Sasaki Riemannian metric, and $T^1\widetilde{M}$ and then $TT^1\widetilde{M}$ with their induced Riemannian metric. We denote by $\|\cdot\|$ their Riemannian norms. Note that the distance on $T^1\widetilde{M}$ defined by Equation (5) defines the same Hölder structure as the Riemannian distance, by Lemma 2.2.

Given a Euclidean space E with finite dimension n and $k \in \{1, \cdots, n-1\}$, we endow the *Grassmannian manifold* $\mathrm{Gr}_k(E)$ of k-dimensional vector subspaces of E with the distance

$$d_E(A, A') = \min\left\{\|w - w'\| \;:\; w \in A,\; w' \in A',\; \|w\| = \|w'\| = 1\right\}.$$

Let m be the dimension of $\widetilde{M}$. Let $E \to T^1\widetilde{M}$ be a vector bundle over $T^1\widetilde{M}$ of rank $r \geq 1$, endowed with a Riemannian bundle metric and with a length distance inducing the Euclidean distance on each fiber. We will be interested in two such Riemannian vector bundles, the tangent bundle of $T^1\widetilde{M}$ endowed with the Sasaki metric and its Riemannian distance, and the fiber bundle over $T^1\widetilde{M}$ whose fiber over $v \in T^1\widetilde{M}$ is the product Euclidean space $T_vT^1\widetilde{M} \times T_{\phi_1 v}T^1\widetilde{M}$, with the distance induced by the Riemannian distance on $TT^1\widetilde{M} \times TT^1\widetilde{M}$. For $k \in \mathbb{N}$, denote by $\mathrm{Gr}_k(E) \to T^1\widetilde{M}$ the *Grassmannian bundle* of rank k of $E \to T^1\widetilde{M}$, whose fiber over $v \in T^1\widetilde{M}$ is the Grassmannian manifold $\mathrm{Gr}_k(E_v)$ of k-dimensional vector subspaces of the fiber E_v of $E \to T^1\widetilde{M}$ over v. For instance, $v \mapsto E^{ss}(v)$ and $v \mapsto E^{su}(v)$ are continuous sections of $\mathrm{Gr}_{m-1}(TT^1\widetilde{M}) \to T^1\widetilde{M}$. Upon identifying a linear map and its graph, the map $v \mapsto T_v\phi_1$ is a continuous section of the

Grassmannian bundle of rank $2m-1$ of the second Riemannian vector bundle above. Since the unit spheres of the elements of $\mathrm{Gr}_k(E)$ are nonempty compact subsets of the metric space E if $1 \le k \le r$, we endow $\mathrm{Gr}_k(E)$ with the distance d, where $d(A, A')$ is the Hausdorff distance between the unit spheres of the k-dimensional Euclidean spaces $A, A' \in \mathrm{Gr}_k(E)$. The restriction of d to each fiber $\mathrm{Gr}_k(E_v)$ is the distance d_{E_v} defined above.

Theorem 7.3 *Let $\widetilde{M}$ be a complete simply connected Riemannian manifold, with dimension at least 2 and pinched negative sectional curvature. Assume that the derivatives of the sectional curvature are uniformly bounded. Then the maps $v \mapsto E^{su}(v)$ and $v \mapsto E^{ss}(v)$ defined on $T^1\widetilde{M}$ are (globally) Hölder-continuous.*

The assumption that the derivatives of the sectional curvature are uniformly bounded is necessary: Ballmann-Brin-Burns [BaBB] have constructed a finite volume complete Riemannian surface with pinched negative sectional curvature whose strong stable foliation is not Hölder-continuous.

Since the distributions are Lipschitz (uniformly smooth) in the direction of the flow lines, note that the maps $v \mapsto E^u(v)$, $v \mapsto E^s(v)$ are also (uniformly locally) Hölder-continuous, and that the regularity of $\widetilde{F}^{su}$ claimed in Theorem 7.1 holds. The reversibility of the potential F^{su} follows from the fact that the Liouville measure is invariant under the geodesic flow. The boundedness of $\widetilde{F}^{su}$ follows from the bounds on the sectional curvature and its tangent map.

We will only prove the result for $v \mapsto E^{ss}(v)$, giving furthermore an explicit estimate on the Hölder exponent. The strong unstable case is similar, by time reversal.

Let us give a few historical comments on Theorem 7.3. When $\widetilde{M}$ has dimension 2 and has a cocompact discrete isometry group, the strong stable and unstable foliations are C^1, hence Hölder regular by compactness (see [Hop2]). Furthermore, Hurder and Katok [HuK, Theo. 3.1, Coro. 3.5] have proved that these subbundles are $\mathrm{C}^{1,\alpha}$ for every $\alpha \in \,]0,1[$ (see also [HiP]), and that if they are $\mathrm{C}^{1,1}$, then they are C^∞, in which case by a theorem of Ghys [Ghy, p. 267], the geodesic flow is C^∞-conjugated to the geodesic flow of a hyperbolic surface.

When $\widetilde{M}$ has dimension at least 3 and has a cocompact discrete isometry group, this result of Hölder regularity of the strong stable and unstable foliations is due to Anosov [Ano]. We follow Brin's and Brin-Stuck's proofs in [Bri], which modernises Anosov's original proof under the same assumptions, and [BrS, §6.1] in the case of Anosov diffeomorphism (not flows) on compact manifolds. Here, we provide details to explain why this proof works under the assumptions of Theorem 7.3.

Proof of Theorem 7.3. Recall that the geodesic flow of $\widetilde{M}$ is an *Anosov flow* for the Sasaki metric: there exist $c > 0$ and $\lambda \in \,]0,1[$ (and one may take $\lambda = e^{-1}$ since the sectional curvature of $\widetilde{M}$ is normalised to have upper bound -1, by standard arguments of Jacobi fields) such that for all $v \in T^1\widetilde{M}$, $V^{ss} \in E^{ss}(v)$, $V^{su} \in E^{su}(v)$ and $t \ge 0$, we have

$$\|T\phi_t(V^{ss})\| \le c\,\lambda^t\,\|V^{ss}\| \quad \text{and} \quad \|T\phi_{-t}(V^{su})\| \le c\,\lambda^t\,\|V^{su}\|\,,$$

and for all $v \in T^1\widetilde{M}$, $V^s \in E^s(v)$, $V^u \in E^u(v)$ and $t \ge 0$, we have

$$\|T\phi_{-t}(V^s)\| \ge c^{-1}\|V^s\| \quad \text{and} \quad \|T\phi_t(V^u)\| \ge c^{-1}\|V^u\|\,.$$

We start by giving two lemmas, the second one, that we state without proof, involving only linear algebra.

Lemma 7.4 *Let $\widetilde{M}$ be a complete simply connected Riemannian manifold, with dimension at least 2 and pinched negative sectional curvature. Then the angles between $E^{ss}(v)$ and any one of $E^{su}(v)$, $E^0(v)$ or $E^u(v)$ are uniformly bounded from below by a positive constant.*

Proof. We prove that the angle between $E^{ss}(v)$ and $E^u(v)$ is bounded from below by a positive constant, the other cases are analogous.

Using the operator norm on linear maps, let $C = \sup_{v\in T^1\widetilde{M}} \|T_v\phi_1\|$, which is finite since M has pinched negative curvature (see for instance [Bal, page 65]). Let $c > 0$ and $\lambda \in \,]0, 1[$ be as above. Let $N \in \mathbb{N}$ be large enough so that $c^{-1} \geq 2c\lambda^N$. For all $v \in T^1\widetilde{M}$, $V^{ss} \in E^{ss}(v)$ and $V^u \in E^u(v)$ with $\|V^{ss}\| = \|V^u\| = 1$, we have

$$\begin{aligned} C^N\|V^u - V^{ss}\| &\geq \|T\phi_N(V^u - V^{ss})\| \geq \|T\phi_N(V^u)\| - \|T\phi_N(V^{ss})\| \\ &\geq c^{-1}\|V^u\| - c\lambda^N\|V^{ss}\| \geq c\lambda^N\ . \end{aligned}$$

The result follows. □

Lemma 7.5 *(Brin-Stuck [BrS, Lem. 6.1.1]) Let $c > 0$, $\lambda \in \,]0, 1[$, $\delta \in [0, 1[$ and $D \geq 1$, let E be a finite dimensional Euclidean vector space, let A^1, A^2 be two vector subspaces of E of the same dimension, and let $(L^1_n)_{n\in\mathbb{N}}, (L^2_n)_{n\in\mathbb{N}}$ be two sequences of linear endomorphisms of E such that, for every $n \in \mathbb{N}$, for $i = 1, 2$, for all $v \in A^i$ and $w \in (A^i)^\perp$, we have*

$$\|L^1_n - L^2_n\| \leq \delta D^n, \quad \|L^i_n(v)\| \leq c\,\lambda^n\,\|v\|, \quad \text{and} \quad \|L^i_n(w)\| \geq c^{-1}\|w\|\ .$$

Then $d_E(A^1, A^2) \leq 3\,\frac{c^2}{\lambda}\,\delta^{\,\frac{-\log\lambda}{\log D - \log\lambda}}$.

Up to replacing the Sasaki metric by the equivalent (by Lemma 7.4) Riemannian metric whose norm is

$$\|V\|' = \sqrt{\|\,V^{su}\,\|^2 + \|\,V^0\,\|^2 + \|\,V^{ss}\,\|^2} \tag{90}$$

for every $V = V^{su} + V^0 + V^{ss}$ where $V^{su} \in E^{su}(v)$, $V^0 \in E^0(v)$ and $V^{ss} \in E^{ss}(v)$, we assume that the subspaces $E^{su}(v), E^0(v)$ and $E^{ss}(v)$ are pairwise orthogonal for every $v \in T^1\widetilde{M}$. For all $v, w \in T^1\widetilde{M}$ and $W \in T_wT^1M$, we denote by $\|^v_w\, W$ the parallel transport of W along any fixed geodesic segment from w to v. Note that the Sasaki metric on $T^1\widetilde{M}$ has a positive lower bound on its injectivity radius at all points, and this remains true with the modified metric. In particular, there exists $\kappa > 0$ such that for all $v, w \in T^1\widetilde{M}$, if $d(v, w) < \kappa$, there exists a unique geodesic segment from w to v. Since $\widetilde{M}$ has pinched negative curvature, there exists $\kappa' \in \,]0, \kappa[$ such that for all $v, w \in T^1\widetilde{M}$ with $d(v, w) < \kappa'$, we have $d(\phi_1 v,\ phi_1 w) < \kappa$. Let $D \geq 2$ be a constant such that, for all $v, w \in T^1\widetilde{M}$,

$$\|T_v\phi_1\| \leq \frac{D}{2} \quad \text{and} \quad \big\|\ T_v\phi_1 - \|^{\phi_1 v}_{\phi_1 w} \circ T_w\phi_1 \circ \|^w_v\ \big\| \leq D\ d(v, w)\ ,$$

which exists since $v \mapsto T_v\phi_1$ is bounded and Lipschitz, since the derivatives of the sectional curvature are uniformly bounded (see for instance [Bal, page 64]). Let

$$a_n = \|T_v\phi_n\| \quad \text{and} \quad b_n = \big\|\ T_v\phi_n - \|^{\phi_n v}_{\phi_n w} \circ T_w\phi_n \circ \|^w_v\ \big\|\ .$$

Then $a_{n+1} = \|T_{\phi_n v}\,\phi_1 \circ T_v\phi_n\| \le a_1 a_n$ and

$$\begin{aligned} b_{n+1} =& \|T_{\phi_n v}\,\phi_1 \circ T_v\phi_n - \|_{\phi_1\phi_n w}^{\phi_1\phi_n v} \circ T_{\phi_n w}\phi_1 \circ \|_{\phi_n v}^{\phi_n w} \;(\;\|_{\phi_n w}^{\phi_n v} \circ T_w\phi_1 \circ \|_v^w\;)\| \\ &\le a_1 b_n + b_1 a_n \;. \end{aligned}$$

Hence by induction, $a_n \le \frac{D}{2}^n$ and $b_n \le D^n d(v,w)$.

Now, let $v, w \in T^1\widetilde{M}$ be such that $d(v,w) < \kappa'$. For every $n\ in\mathbb{N}$, let $f_{v,\,n} : T_{\phi_n v}T^1\widetilde{M} \to T_vT^1\widetilde{M}$ be an isometric map. We are going to apply Lemma 7.5 with c, λ given by Anosov's flow property, with D as above, with $\delta = d(v,w)$, with E the Euclidean space $T_vT^1\widetilde{M}$, and with

$$A^1 = E^{ss}(v), \quad A^2 = \|_w^v\, E^{ss}(w), \quad L_n^1 = f_{v,\,n} \circ T_v\phi_n, \quad L_n^2 = f_{v,\,n}\circ \|_{\phi_n w}^{\phi_n v} \circ T_w\phi_n \circ \|_v^w \;,$$

for every $n \in \mathbb{N}$. The hypotheses of Lemma 7.5 are easily checked (since $(A^1)^\perp = E^u(v)$ and the parallel transport is isometric), and we hence have

$$d(E^{ss}(v), \|_w^v\, E^{ss}(w)) \le 3\,\frac{c^2}{\lambda}\; d(v,w)^{\frac{-\log\lambda}{\log D - \log\lambda}} \;.$$

Since $d(\,\|_w^v\, E^{ss}(w), E^{ss}(w)) \le d(v,w)$ by Equation (3), the result hence follows by the triangle inequality. □

7.2 The Liouville measure as an equilibrium state

In this subsection, we recall the definition of Lyapounov's exponents and the main results about them (Oseledets's theorem, Ruelle's inequality, the Pesin formula, see for instance [Man2, Led1, Man1, KaM]), in order to prove that the Liouville measure, once normalised, is the equilibrium state of an appropriate potential, under compactness assumptions.

We denote here by $\phi = \phi_1$ the time one map of the geodesic flow $(\phi_t)_{t\in\mathbb{R}}$ on T^1M. We fix a probability measure μ on T^1M invariant under $(\phi_t)_{t\in\mathbb{R}}$. Recall that $\log^+ t = \max\{\log t, 0\}$ for every $t > 0$, and that $T^1\widetilde{M}$ is endowed with Sasaki's Riemannian metric.

Oseledets's theorem (see for instance [Ose], [KaM, Theo. S.2.9 page 665]) asserts that if $\int_{T^1M} \log^+ \|T\phi^{\pm 1}\|\, d\mu < \infty$ (in particular if M is compact), then there exists a Borel subset Reg of T^1M, invariant under $(\phi_t)_{t\in\mathbb{R}}$ and with full measure with respect to μ, such that for every $v \in$ Reg, there exists a unique direct sum decomposition (of closed convex cones when Γ has torsion), called the *Oseledets decomposition* at v,

$$T_vT^1M = \oplus_{i=1}^{s(v)} E_i(v)$$

and unique real numbers $\chi_1(v) < \cdots < \chi_{s(v)}(v)$, called the *Lyapounov exponents* of v, such that

(1) the map $v \mapsto s(v)$ from Reg to $\mathbb{N}$ is measurable; for all $i, r \in \mathbb{N}$, the map $v \mapsto E_i(v)$ from $\{v \in \text{Reg} \;:\; s(v) \ge i, \dim E_i(v) = r\}$ to the space of closed subsets of TT^1M (endowed with Chabauty's topology) is measurable; and for all $v \in$ Reg and $i \in \{1, \dots, s(v)\}$, we have $T_v\phi(E_i(v)) = E_i(\phi v)$ and $\chi_i(\phi v) = \chi_i(v)$;

(2) for all $v \in$ Reg, $i \in \{1, \dots, s(v)\}$ and $V \in E_i(v) - \{0\}$,

$$\lim_{n\to\pm\infty} \frac{1}{n} \log \|T_v\phi^n(V)\| = \chi_i(v) \;;$$

(3) for all $v \in \mathrm{Reg}$ and $i \in \{1, \dots, s(v)\}$, if $\dim E_i(v) = k_i(v)$, then

$$\lim_{n \to +\infty} \frac{1}{n} \log J^{E_i} \phi^n(v) = k_i(v)\, \chi_i(v)\,,$$

where $J^{E_i}\phi^n(v)$ denotes the Jacobian of the restriction of $T_v\phi^n$ to $E_i(v)$;

(4) the angle between two subspaces of the Oseledets decomposition has at most subexponential decay: for all v in Reg, $i \neq j$ in $\{1, \dots, s(v)\}$, $V \in E_i(v) - \{0\}$ and $W \in E_j(v) - \{0\}$, we have

$$\lim_{n \to \pm\infty} \frac{1}{|n|} \log \big| \sin \angle(T_v\phi^n(V), T_v\phi^n(W))\big| = 0\,.$$

In particular, since $E^{su}(v) = \oplus_{1 \leq i \leq s(v),\ \chi_i(v) > 0}\, E_i(v)$ for every $v \in \mathrm{Reg}$, we deduce from the last two points that for μ-almost every $v \in T^1M$, we have

$$\lim_{n \to +\infty} \frac{1}{n} \int_0^n F^{su}(\phi_t v)\, dt = - \lim_{n \to +\infty} \frac{1}{n} \log J^{su}(v, n) = - \sum_{1 \leq i \leq s(v),\ \chi_i(v) > 0} k_i(v)\, \chi_i(v)\,.$$

Note that when μ is ergodic for $(\phi_t)_{t \in \mathbb{R}}$, by Birkhoff's ergodic theorem, for μ-almost every $v \in T^1M$, this equality becomes

$$\int_{T^1M} F^{su}\, d\mu = - \sum_{1 \leq i \leq s(v),\ \chi_i(v) > 0} k_i(v)\, \chi_i(v)\,. \tag{91}$$

Ruelle's inequality (see [Rue1]) asserts that if M is compact, then

$$h_\mu(\phi) \leq \int_{v \in T^1M} \sum_{1 \leq i \leq s(v),\ \chi_i(v) > 0} k_i(v)\, \chi_i(v)\, d\mu(v)\,. \tag{92}$$

It is unclear if it is possible to extend the proof of Ruelle's inequality in the case where M is noncompact, with perhaps additional geometric assumptions besides the pinched negative sectional curvature.

When M is compact and μ is ergodic, by Equation (91) and Equation (92), we have $h_\mu(\phi) + \int_{T^1M} F^{su}\, d\mu \leq 0$. Since the upper bound defining the topological pressure of a potential (see Chapter 6) may be taken on the ergodic probability measures invariant under $(\phi_t)_{t \in \mathbb{R}}$ by the convexity properties of the metric entropy, we hence have

$$P(\Gamma, F^{su}) \leq 0\,. \tag{93}$$

The *Pesin formula* (see for instance [Pes, Man1]) asserts that if μ is absolutely continuous with respect to the Lebesgue measure class, then we have equality in Equation (92):

$$h_\mu(\phi) = \int_{v \in T^1M} \sum_{1 \leq i \leq s(v),\ \chi_i(v) > 0} k_i(v)\, \chi_i(v)\, d\mu(v)\,. \tag{94}$$

If M is noncompact, it is unclear when the Pesin formula remains valid.

In particular, if M is compact, then the Liouville measure $\operatorname{vol}_{T^1M}$ is ergodic (see for instance [Bri, page 95]), and the probability measure L proportional to the Liouville measure satisfies

$$P(\Gamma, F^{su}) \leq 0 = h_L(\phi) + \int_{T^1M} F^{su}\, dL\,.$$

Hence L is an equilibrium state for the potential F^{su}. The variational principle (Theorem 6.1) implies that $L = m_{F^{su}}$, which proves the following expected result.

Theorem 7.6 *If M is compact, then the normalised Liouville measure $\frac{\operatorname{vol}_{T^1M}}{\operatorname{Vol}(T^1M)}$ coincides with the normalised Gibbs measure $\frac{m_{F^{su}}}{\|m_{F^{su}}\|}$ of the potential F^{su}.* $\square$

Remark. Assume in this remark that M has finite volume with constant curvature. Then F^{su} is constant, hence the Gibbs measure of the potential F^{su} and the Bowen-Margulis measure (which is the Gibbs measure of the potential 0) coincide. The Patterson-Sullivan measure of Γ at the origin of the ball model of the real hyperbolic n-space $\mathbb{H}^n_{\mathbb{R}}$ may be taken to be the standard Riemannian measure of the unit sphere $\mathbb{S}^{n-1} = \partial_\infty \mathbb{H}^n_{\mathbb{R}}$. The exact ratio between the total masses of T^1M for the Liouville measure and for the Gibbs measure $m_{F^{su}}$ is computed in [PaP3, Sect. 7], yielding

$$\frac{\|m_{F^{su}}\|}{\operatorname{Vol}(T^1M)} = 2^{n-1}\operatorname{Vol}(\mathbb{S}^{n-1})\,.$$

7.3 The Liouville measure satisfies the Gibbs property

In this subsection, we prove that, under the previous bounded geometry assumption, the Liouville measure satisfies a Gibbs property as defined in Subsection 3.8, and we deduce from this our criterion (Theorem 7.2) for the Liouville measure to be a Gibbs measure.

Proposition 7.7 *Assume that the derivatives of the sectional curvature are uniformly bounded. Then the Liouville measure on T^1M satisfies the Gibbs property for the potential F^{su} and the constant $c(F^{su}) = 0$.*

Proof. Note that F^{su} is Hölder-continuous and bounded by Theorem 7.1.

Using ideas of Bowen-Ruelle in [BoR, Lem. 4.2], we follow the proof of Katok-Hasselblatt in [KaH, Lem. 20.4.2], by adapting its arguments for diffeomorphisms to the case of flows and by taking care of the noncompactness of T^1M.

Fix a compact subset $\widetilde{K}$ of $T^1\widetilde{M}$. Note that by the definition of the dynamical balls in Subsection 3.8, for all $r \in\,]0,1]$, $v \in T^1\widetilde{M}$ and $T, T' \geq 1$ with $\phi_{-T'}v \in \widetilde{K}$, the set $\phi_{-T'}B(v;T,T',r)$ is contained in the compact subset $\pi^{-1}(\mathscr{N}_1(\pi(\widetilde{K})))$ of $T^1\widetilde{M}$. Hence the multiplicity of the restriction to $\phi_{-T'}B(v;T,T',r)$ of the map $T^1\widetilde{M} \to T^1M$ is bounded by a constant depending only on $\widetilde{K}$, by the discreteness of the action of Γ on $T^1\widetilde{M}$.

Since the Liouville measure on $T^1\widetilde{M}$ is invariant under the geodesic flow and by the definition 3.14 (in Subsection 3.8) of the Gibbs property, we hence only have to prove that there exist $r \in\,]0,1]$, and a constant $C_{\widetilde{K},r} \geq 1$ such that for all $v \in T^1\widetilde{M}$ and $T, T' \geq 1$ with $\phi_T v \in \widetilde{K}$ and $\phi_{-T'}v \in \Gamma\widetilde{K}$, we have

$$\frac{1}{C_{\widetilde{K},r}}\, e^{\int_{-T'}^{T} \widetilde{F}^{su}(\phi_t v)\, dt} \leq \operatorname{vol}_{T^1\widetilde{M}}(B(v;T,T',r)) \leq C_{\widetilde{K},r}\, e^{\int_{-T'}^{T} \widetilde{F}^{su}(\phi_t v)\, dt}\,. \tag{95}$$

We may restrict T and T' to being nonzero integers: The result for general T and T' follows by using the inclusion properties of the dynamical balls, by replacing $\widetilde{K}$ by the larger compact set $\cup_{t\in[-1,1]}\phi_t\widetilde{K}$ and by modifying $C_{K,r}$, since $\widetilde{F}^{su}$ is bounded. We denote by ϕ the time one map of the geodesic flow in this subsection.

The scheme of the proof is the following one. Using the stable and unstable foliations, we define a local product structure into stable leaves and strong unstable ones near points of $\widetilde{K}$ and modify slightly the dynamical balls to adapt them to this local product structure. We then use Fubini's theorem to control the volume of the dynamical balls: The integral along the stable leaves is uniformly bounded; the integral along the unstable leaves is controlled, using a change of variable, by the unstable Jacobian J^{su} of the geodesic flow, which is related to the integral of the potential F^{su} by the definition of F^{su}.

We use the Riemannian metric on $T^1\widetilde{M}$ whose norm is the norm $\|\cdot\|'$ defined in Equation (90). This metric is Γ-invariant and, for every $v\in T^1\widetilde{M}$, the tangent subspaces $E^{su}(v), E^0(v), E^{ss}(v)$ of $T_vT^1\widetilde{M}$ become pairwise orthogonal, hence the Lebesgue measure on $T_vT^1\widetilde{M}$ is the product of the Lebesgue measures on $E^{su}(v)$ and $E^0(v)+E^{ss}(v)$. For all $v\in T^1\widetilde{M}$ and $r>0$, we denote by $B_v(0,r)$ the open ball of center 0 and radius r in $T_vT^1\widetilde{M}$ for this norm.

Lemma 7.8 *There exists $\varepsilon=\varepsilon_{\widetilde{K}}>0$ such that for every $v\in\Gamma\widetilde{K}$, there exists a C^1 chart $\varphi_v : B_v(0,\varepsilon)\subset T_vT^1\widetilde{M}\to V_v=\varphi_v(B_v(0,\varepsilon))\subset T^1\widetilde{M}$ such that $\varphi_v(0)=v$, $\varphi_{\gamma v}=\varphi_v$ for every $\gamma\in\Gamma$, and*

$$\varphi_v(E^s(v)\cap B_v(0,\varepsilon))\subset W^s(v)\quad and\quad \varphi_v(E^u(v)\cap B_v(0,\varepsilon))\subset W^u(v)\,.$$

Moreover, the map $\Phi_v=\varphi_{\phi v}^{-1}\circ\phi\circ\varphi_v : T_vT^1\widetilde{M}\to T_{\phi v}T^1\widetilde{M}$ satisfies $d(\Phi_v)_0=T_v\phi$.

Given $v\in\Gamma\widetilde{K}$ and $w\in V_v$, we will denote by w^{su}, w^0, w^{ss} the components of $\varphi_v^{-1}(w)$ in the direct sum decomposition $T_vT^1\widetilde{M}=E^{su}(v)\oplus E^0(v)\oplus E^{ss}(v)$, as well as $w^s=w^0+w^{ss}$.

Proof. This is due to Bowen-Ruelle [BoR, Appendix A.1], see also [KaH, Sect. 19.1.d], when M is compact and Γ torsion free. The proof extends, the only difference is that the parameter $\varepsilon>0$ is uniform on $\Gamma\widetilde{K}$, but possibly not on the whole space $T^1\widetilde{M}$ by the lack of bounded geometry properties which would be ensured by the compactness of T^1M. $\Box$

We now replace the dynamical balls by sets better adapted to the above local product structure. For all $r>0$, $v\in T^1\widetilde{M}$ and $n,n'\in\mathbb{N}-\{0\}$, consider the sets

$$\begin{aligned}C(v;n,n',r)=\big\{w\in T^1\widetilde{M}\ :\ &\phi^n w\in V_{\phi^n v},\ \ \phi^{-n'}w\in V_{\phi^{-n'}v},\\ &\max_{k\in\{-n',n\}}\max\big\{\|(\phi^k w)^{ss}\|',\|(\phi^k w)^{su}\|',\|(\phi^k w)^0\|'\big\}\ <r\big\}\,.\end{aligned}$$

Note that $\gamma C(v;n,n',r)=C(\gamma v;n,n',r)$ for every $\gamma\in\Gamma$, and that $\phi^{-n'}C(v;n,n',\rho)$ is contained in the image $V_{u'}$ of $\varphi_{u'}$ for $u'=\phi^{-n'}v$

Lemma 7.9 *There exist $r_0,c_0>0$ such that for all $r\in\,]0,r_0]$, $n,n'\geq 1$ and $v\in T^1\widetilde{M}$ such that $\phi^n v,\phi^{-n'}v\in\Gamma\widetilde{K}$, we have*

$$C(v;n,n',\frac{r}{c_0})\subset B(v;n,n',r)\subset C(v;n,n',c_0r)\,.$$

Proof. Recall that Sasaki's distance d_S on $T^1\widetilde{M}$ is equivalent to the distance defined by $\|\cdot\|'$. Hence, there exists a constant $c_1 > 0$ depending only on $\widetilde{K}$ such that for all $v \in \Gamma\widetilde{K}$ and $w \in V_v$, we have

$$\frac{1}{c_1}\max(\|w^{ss}\|', \|w^{su}\|', \|w^0\|') \leq d_S(v,w) \leq c_1 \max(\|w^{ss}\|', \|w^{su}\|', \|w^0\|') \,.$$

By convexity, for all $v, w \in T^1\widetilde{M}$ and $n, n' \in \mathbb{N} - \{0\}$, we have

$$\begin{aligned}&\max_{t\in[-n',\,n]} d_{\widetilde{M}}\big(\pi(\phi_t v), \pi(\phi_t w)\big)\\ =\;& \max\big\{\max_{t\in[-n',\,-n'+1]} d_{\widetilde{M}}\big(\pi(\phi_t v), \pi(\phi_t w)\big),\; \max_{t\in[n-1,\,n]} d_{\widetilde{M}}\big(\pi(\phi_t v), \pi(\phi_t w)\big)\big\}\,.\end{aligned}$$

As already mentioned in Subsection 2.3, by for instance [Bal, page 70], there exists hence $c_3 > 0$ such that for all $v, w \in T^1\widetilde{M}$ and $n, n' \in \mathbb{N} - \{0\}$, we have

$$\frac{1}{c_2}\max_{k\in\{-n',\,n\}} d_S(\phi^k v, \phi^k w) \leq \max_{t\in[-n',\,n]} d_{\widetilde{M}}\big(\pi(\phi_t v), \pi(\phi_t w)\big) \leq c_2 \max_{k\in\{-n',\,n\}} d_S(\phi^k v, \phi^k w)\,.$$

Let $r_0 > 0$ be small enough so that, for every $v \in \Gamma\widetilde{K}$, the ball $B_{d_S}(v, c_2 r_0)$ is contained in V_v (which exists by the compactness of $\widetilde{K}$ and equivariance). By the definition of the dynamical balls in Subsection 3.8, the result holds for $c_0 = c_1 c_2$. $\square$

Given $v \in \Gamma\widetilde{K}$, the Liouville measure, which is a smooth measure, is equivalent on V_v to the image by φ_v of the product of the volumes on $E^{su}(v)$, $E^0(v)$, $E^{ss}(v)$, and its density is bounded from above and bounded away from 0 on V_v. Moreover, these bounds can be chosen uniformly in $v \in \Gamma\widetilde{K}$.

Similarly, by Lemma 7.8, for all $v \in T^1\widetilde{M}$ and $n, n' \geq 1$ such that $\phi^n v \in \Gamma\widetilde{K}$ and $u' = \phi^{-n'}v \in \widetilde{K}$, at every point of $B_{u'}(0,\varepsilon)$, the ratio of the Jacobian of $\phi^{n+n'}$ by the Jacobian of $(\Phi_{u'})^{n+n'}$ is bounded from above and below by a positive constant, depending on the Jacobian of the chart maps $\varphi_{u''}$ for $u'' \in \widetilde{K}$ (which is compact). Therefore, up to positive multiplicative constants (depending only on $\widetilde{K}$), we may replace the study of $\mathrm{vol}_{T^1\widetilde{M}}(B(v;n,n',r)) = \mathrm{vol}_{T^1\widetilde{M}}\big(\phi^{-n'}B(v;n,n',r)\big)$ for $r > 0$ small enough by the study of

$$\mathrm{vol}_{T_{u'}T^1\widetilde{M}}\big({\varphi_{u'}}^{-1}(\phi^{-n'}C(v;n,n',\rho))\big)\,,$$

for $\rho > 0$ small enough.

Let $\rho_0 > 0$ be such that for every $u \in \Gamma\widetilde{K}$, the ball $B(u,\rho_0)$ for the Sasaki metric in $T^1\widetilde{M}$ is contained in V_u, which exists by compactness of $\widetilde{K}$ and by equivariance. Let $\rho \in\,]0,\rho_0]$, $n, n' \in \mathbb{N} - \{0\}$ and $v \in T^1\widetilde{M}$ be such that $u' = \phi^{-n'}v \in \Gamma\widetilde{K}$. Observe that $\phi^{-n'}C(v;n,n',\rho)$ contains the local stable manifold $W^s_\rho(u') = W^{ws}(u') \cap B(u',\rho)$. Note that if $w \in W^s_\rho(u')$, then $w^{su} = 0$ and $w = \varphi_{u'}(w^s)$ by the definition of the local coordinates w^{su} and $w^s = w^0 + w^{ss}$ in $T_{u'}T^1\widetilde{M}$ of w. Furthermore, if $w \in \phi^{-n'}C(v;n,n',\rho)$, then $\varphi_{u'}(w^s) \in W^s_\rho(u')$.

From now on, we identify $w \in V_{u'}$ with its image in $T_{u'}T^1\widetilde{M}$ by ${\varphi_{u'}}^{-1}$. We denote by m the dimension of $\widetilde{M}$ and, for every $k \in \mathbb{R}$, by vol_k the k-dimensional Lebesgue measure on any Euclidean space of dimension k. For every $y \in E^s(u')$, let

$$U_y = \{w \in \phi^{-n'}C(v;n,n',\rho)\;:\; w^s = y\}\,.$$

Then by Fubini's theorem

$$\operatorname{vol}_{2m-1}(\phi^{-n'}C(v;n,n',\rho)) = \int_{y\in W^s_\rho(u')} \operatorname{vol}_{m-1}(U_y)\, d\operatorname{vol}_m(y)\,.$$

Note that $\operatorname{vol}_m(W^s_\rho(u'))$ is controlled by uniform constants depending only on ρ and $\widetilde{K}$, and not on u' in $\widetilde{K}$. By the change of variable formula, we have

$$\operatorname{vol}_{m-1}(U_y) = \int_{\phi^{n+n'}U_y} \operatorname{Jac}(\phi^{-n-n'})_{|T\phi^{n+n'}(U_y)}\, d\operatorname{vol}_{m-1}\ .$$

By the definition of U_y and as in [KaH, page 635], there exists a constant $c>0$ depending only on ρ and $\widetilde{K}$ such that

$$\frac{1}{c} \leq \operatorname{vol}_{m-1}(\phi^{n+n'}U_y) \leq c\ .$$

Using the Hölder-continuity of the strong unstable distribution (given by Theorem 7.3), the arguments of the end of the proof of [KaH, Lem. 20.4.2] show that if ρ is small enough, there exists a constant $c>0$ depending only on ρ and $\widetilde{K}$ such that for all $y\in W^s_\rho(u')$,

$$\frac{1}{c}\ \operatorname{Jac}\big({\phi^{-n-n'}}_{|W^{su}(\phi^n v)}\big) \leq \operatorname{Jac}(\phi^{-n-n'})_{|T\phi^{n+n'}(U_y)} \leq c\ \operatorname{Jac}\big({\phi^{-n-n'}}_{|W^{su}(\phi^n v)}\big)\ .$$

Note that $\operatorname{Jac}\big({\phi^{-n-n'}}_{|W^{su}(\phi^n v)}\big) = \widetilde{J}^{su}(\phi^n v, -n-n')$ by the definition of the unstable Jacobian $\widetilde{J}^{su}$. Using the cocycle formula (88), we have by a classical argument

$$\begin{aligned}\int_{-n'}^{n} \widetilde{F}^{su}(\phi_t v)\,dt &= \int_{-n'-n}^{0} \widetilde{F}^{su}(\phi_t\phi^n v)\,dt = \int_{-n'-n}^{0} -\frac{d}{ds}_{|s=0} \log \widetilde{J}^{su}(\phi_t\phi^n v, s)\,dt\\ &= \int_{-n'-n}^{0} -\frac{d}{dt}\log\widetilde{J}^{su}(\phi^n v,t)\,dt = \log\widetilde{J}^{su}(\phi^n v, -n-n')\ .\end{aligned}$$

The result follows. □

We now use the Gibbs property of the Liouville measure $\operatorname{vol}_{T^1M}$ to prove Theorem 7.2.

Proof of Theorem 7.2. By Theorem 7.1 and the assumptions of Theorem 7.2, the potential $\widetilde{F}^{su}$ is Hölder-continuous and bounded. Thus, there exists indeed a Gibbs measure $m_{F^{su}}$ on T^1M for the potential F^{su} (see Subsection 3.7).

Theorem 7.2 will easily follow from the following result.

Lemma 7.10 *Under the assumptions of Theorem 7.2, the Liouville measure* $\operatorname{vol}_{T^1M}$ *is absolutely continuous with respect to* $m_{F^{su}}$*.*

Proof. Since $T^1\widetilde{M}$ is σ-compact and by Γ-equivariance, we only have to prove that for every compact subset K of $T^1\widetilde{M}$, the restriction of $\operatorname{vol}_{T^1\widetilde{M}}$ to K is absolutely continuous with respect to the restriction of $\widetilde{m}_{F^{su}}$ to K. Let us fix a compact subset K of $T^1\widetilde{M}$.

Proposition 3.15 and Proposition 7.7, as well as the comment following Definition 3.14, imply that for every $r>0$, there exists $c_r>0$ (depending only on r and K) and $T_r\geq 0$ such that for all $T,T'\geq T_r$ and $v\in T^1\widetilde{M}$ such that $\phi_{-T'}v, \phi_T v\in\Gamma K$, we have

$$\operatorname{vol}_{T^1\widetilde{M}}(B(v;T,T',r)) \leq c_r\ e^{\int_{-T'}^{T}\widetilde{F}^{su}(\phi_t v)\,dt} \tag{96}$$

and, as $\delta_{\Gamma,\,F^{su}} \le 0$ by the assumptions of Theorem 7.2,

$$\widetilde{m}_{F^{su}}(B(v;T,T',r)) \geq \frac{1}{c_r}\; e^{\int_{-T'}^{T}(\widetilde{F}^{su}(\phi_t v)-\delta_{\Gamma,\,F^{su}})\,dt} \geq \frac{1}{c_r}\; e^{\int_{-T'}^{T}\widetilde{F}^{su}(\phi_t v)\,dt}\;. \tag{97}$$

Let us fix $r>0$. Let B be a measurable subset of K, which in particular has finite Liouville measure. Let B' be the measurable subset of $v\in B$ such that there exists an increasing sequence $(n_k)_{k\in\mathbb{N}}$ in $\mathbb{N}$ such that $\phi_{-n_k}v\in B'$ and $\phi_{n_k}v\in B'$. By the point 2) of the proposition at the end of the introduction in [AaN], since the geodesic flow is 2-recurrent for the Liouville measure under the assumptions of Theorem 7.2, for every measurable subset A of $T^1\widetilde{M}$, if $\operatorname{vol}_{T^1\widetilde{M}}(A)>0$, then

$$\sum_{n\in\mathbb{N}}\operatorname{vol}_{T^1\widetilde{M}}(\phi_{-n}A\cap A\cap\phi_n A)=+\infty\;.$$

Applying this with A the complementary subset of B' in B, this proves that B' has full Liouville measure in B.

By the strict convexity properties of geodesic segments, for every relatively compact open neighbourhood V of B, for every $v\in B$, there exists $S_v\ge 0$ such that for all $T,T'\ge S_v$, the dynamical ball $B(v;T,T',r)$ is contained in V. For every $v\in B'$, fix $t_v\ge\max\{T_r,T_{2r},S_v\}$ such that $\phi_{-t_v}v,\phi_{t_v}v\in K$, which is possible by the definition of B' (contained in K). Up to enlarging K, we may assume that $t_v\in\mathbb{N}$ for every $v\in B'$. Note that the family $(B_{v,\,r}=B(v;t_v,t_v,r))_{v\in B'}$ covers B' and that its union is contained in V.

By a Vitali type of argument, there exists a finite or countable subset I of B' such that the subfamily $(B_{v_i,\,r})_{i\in I}$ has pairwise disjoint elements, and such that the family $(B_{v_i,\,2r})_{i\in I}$ covers B'.

To prove this, the construction proceeds by induction. If $B'=\emptyset$, take $I'=\emptyset$. Otherwise, take $v_1\in B'$ such that t_{v_1} is minimal. Assume that $v_1,\dots,v_k$ are constructed. Let $v_{k+1}\in B'$ with $t_{v_{k+1}}$ minimal in the set of $v\in B'$ such that $B_{v,\,r}$ does not meet $\bigcup_{1\le i\le k}B_{v_i,\,r}$, if this set is nonempty, otherwise the construction stops at rank k. Since V is relatively compact, the number of possible choices of v_{k+1} is finite, hence the sequence of $t_{v_{k+1}}$, if infinite, tends to $+\infty$. The construction then does give a finite or countable family, since $T^1\widetilde{M}$ is separable. By construction, for every $v\in B'$, the dynamical ball $B_{v,\,r}$ meets $B_{v_i,\,r}$ for some $i\in I$, and $t_v\ge t_{v_i}$. Hence $v\in B_{v_i,\,2r}$ by Lemma 3.13 (2). This proves the existence of I as required.

Now by the σ-additivity of $\widetilde{m}_{F^{su}}$ (the elements of $(B_{v_i,\,r})_{i\in I}$ are pairwise disjoint) and by Equation (97) and Equation (96), we have

$$\begin{aligned}\widetilde{m}_{F^{su}}(V)&\ge\widetilde{m}_{F^{su}}\Big(\bigcup_{v\in B'}B_{v,\,r}\Big)\ge\widetilde{m}_{F^{su}}\Big(\bigcup_{i\in I}B_{v_i,\,r}\Big)=\sum_{i\in I}\;\widetilde{m}_{F^{su}}(B_{v_i,\,r})\\&\ge\frac{1}{c_r}\sum_{i\in I}e^{\int_{-t_{v_i}}^{t_{v_i}}\widetilde{F}^{su}(\phi_t v)\,dt}\ge\frac{1}{c_{2r}c_r}\sum_{i\in I}\operatorname{vol}_{T^1\widetilde{M}}(B_{v_i,\,2r})\\&\ge\frac{1}{c_{2r}c_r}\operatorname{vol}_{T^1\widetilde{M}}\Big(\bigcup_{i\in I}B_{v_i,\,2r}\Big)\ge\frac{1}{c_{2r}c_r}\operatorname{vol}_{T^1\widetilde{M}}(B')=\frac{1}{c_{2r}c_r}\operatorname{vol}_{T^1\widetilde{M}}(B)\;.\end{aligned}$$

As V is any relatively compact open neighbourhood of B, by the regularity of $\widetilde{m}_{F^{su}}$, we hence have $\operatorname{vol}_{T^1\widetilde{M}}(B)\le c_{2r}c_r\;\widetilde{m}_{F^{su}}(B)$ for every Borel subset B of K. This proves the result. $\Box$

Let us now conclude the proof of Theorem 7.2. Recall that $\Omega_c\Gamma$ is the subset of elements of T^1M which are positively and negatively recurrent under the geodesic flow. Since the Liouville measure on T^1M is 2-recurrent hence completely conservative, the measurable set $\Omega_c\Gamma$ has full Liouville measure, the limit set of Γ is equal to $\partial_\infty\widetilde{M}$ and the (topological) nonwandering set $\Omega\Gamma$ of the geodesic flow is equal to T^1M.

Lemma 7.10 implies that $m_{F^{su}}(\Omega_c\Gamma) > 0$, so that $m_{F^{su}}$ is not completely dissipative. By the Hopf-Tsuji-Sullivan-Roblin theorems 5.3 and 5.4, this implies that (Γ, F^{su}) is not of convergence type, hence is of divergence type, and that $m_{F^{su}}$ is ergodic and completely conservative, hence also gives full measure to $\Omega_c\Gamma$. A standard ergodicity argument then implies that vol_{T^1M} and $m_{F^{su}}$ are proportional, as required. Their constants for the Gibbs property have to be equal, which proves (by Theorem 4.4 and by Theorem 6.1 (2) which holds since F^{su} is bounded) the final equalities stated in Theorem 7.2. $\Box$

Remark. The same proof shows that if the derivatives of the sectional curvature of $\widetilde{M}$ are uniformly bounded, if $\delta_{\Gamma,\,F^{su}} = 0$, and if the geodesic flow of M is 2-recurrent with respect to the Gibbs measure $m_{F^{su}}$ on T^1M, then $m_{F^{su}}$ is proportional to the conservative part of the Liouville measure vol_{T^1M}.

Note that the Ahlfors conjecture, proved by Calegari-Gabai [CG] (after works of Bonahon and Canary) says that if $\widetilde{M} = \mathbb{H}^3_{\mathbb{R}}$ is the real hyperbolic space of dimension 3, if Γ is finitely generated, then either $\Lambda\Gamma$ has Lebesgue measure zero, or $\Lambda\Gamma = \partial_\infty\mathbb{H}^3_{\mathbb{R}}$. But when Γ is not finitely generated, it could happen that $\Lambda\Gamma$ and $\partial_\infty\widetilde{M} - \Lambda\Gamma$ are both of positive Lebesgue measure in $\partial_\infty\mathbb{H}^3_{\mathbb{R}}$, so that the conservative and the dissipative part of the Liouville measure vol_{T^1M} would both be nontrivial. This explains how it could happen that $m_{F^{su}}$ could be proportional to the conservative part of the Liouville measure, but not to the Liouville measure.

The same proof also shows the following fact, generalising known results when M is compact (see for instance [KaH, Sect. 20.3]).

Proposition 7.11 *Let $\widetilde{M}$ be a complete simply connected Riemannian manifold, with dimension at least 2 and pinched sectional curvature at most -1 and Γ is a nonelementary discrete group of isometries of $\widetilde{M}$. Let $\widetilde{F} : T^1\widetilde{M} \to \mathbb{R}$ be a Γ-invariant Hölder-continuous potential and let $c \in \mathbb{R}$. Then there exists, up to a multiplicative constant, at most one locally finite (Borel, positive) measure on T^1M invariant under the geodesic flow, ergodic and 2-recurrent, which satisfies the Gibbs property (see Definition 3.14) for the potential F and the constant c.*

8 Finiteness and mixing of Gibbs states

Let $(\widetilde{M}, \Gamma, \widetilde{F})$ be as in the beginning of Chapter 2: $\widetilde{M}$ is a complete simply connected Riemannian manifold, with dimension at least 2 and pinched sectional curvature at most -1; Γ is a nonelementary discrete group of isometries of $\widetilde{M}$; and $\widetilde{F} : T^1\widetilde{M} \to \mathbb{R}$ is a Hölder-continuous Γ-invariant map.

Fix x, y in $\widetilde{M}$. Assume that $\delta_{\Gamma,\,F} < +\infty$ (see Subsection 3.2 for comments on this condition, not so important but without which we can not even define Gibbs measures). Let $\widetilde{m}_F$ be the Gibbs measure on $T^1\widetilde{M}$ associated with a pair of Patterson densities $(\mu^\iota_x)_{x\in\widetilde{M}}$ and $(\mu_x)_{x\in\widetilde{M}}$ for respectively $(\Gamma, F\circ\iota)$ and (Γ, F) of (common) dimension $\delta_{\Gamma,\,F\circ\iota} = \delta_{\Gamma,\,F}$. Denote by $\|m_F\|$ the total mass of the measure m_F on T^1M induced by $\widetilde{m}_F$.

Compared to the previous ones, the sections 9, 10 and 11.7 will require stronger assumptions on the Gibbs measure m_F. We will assume that it is finite and mixing under the geodesic flow. These assumptions are already present in the case $F = 0$ considered by [Rob1]. But essentially, as explained in the coming Subsection 8.1, only the finiteness of m_F is important (it is a nonempty assumption, even in the case $F = 0$, see for instance [DaOP]).

Indeed, we have proved in Corollary 5.14 that, when it is finite, the Gibbs measure m_F is unique up to scaling and is ergodic (and the Patterson densities $(\mu^\iota_x)_{x\in\widetilde{M}}$ and $(\mu_x)_{x\in\widetilde{M}}$ are also unique up to scaling). We will give one finiteness criterion in Subsection 8.2. But we prefer to start by stating the main dynamical tool to be used in the coming chapters, saying that the mixing property is essentially always true as soon as the Gibbs measure m_F is finite.

8.1 Babillot's mixing criterion for Gibbs states

Since any Gibbs measure is a quasi-product measure (see Subsection 3.7 and Subsection 3.9), we may apply Babillot's result [Bab2, Theo. 1] to obtain the mixing property of the geodesic flow. Recall that the *length spectrum* of M is the subset of $\mathbb{R}$ consisting of the translation lengths of the elements of Γ, or, equivalently, of the lengths of the periodic orbits of the geodesic flow on T^1M. A continuous flow of homeomorphisms $(\varphi_t)_{t\in\mathbb{R}}$ of a topological space X is *topologically mixing* if for all nonempty open subsets U and V of X, there exists $t_0 \in \mathbb{R}$ such that $\varphi_t(U) \cap V \neq \emptyset$ for every $t \geq t_0$.

Theorem 8.1 (Babillot) *If $\delta_{\Gamma,\,F} < +\infty$ and if m_F is finite, then the following conditions are equivalent:*

(1) the geodesic flow $(\phi_t)_{t\in\mathbb{R}}$ on T^1M is mixing for the measure m_F;

(2) the geodesic flow $(\phi_t)_{t\in\mathbb{R}}$ on T^1M is topologically mixing on its (topological) nonwandering set $\Omega\Gamma$;

(3) the length spectrum of M is not contained in a discrete subgroup of $\mathbb{R}$.

For instance, the last condition is satisfied if $\widetilde{M}$ is a symmetric space, or if Γ contains a parabolic element, or if M has dimension 2, or if $\Lambda\Gamma$ is not totally disconnected (see for instance [Dal1, Dal2] and their references).

Conjecturally, the non arithmeticity of the length spectrum (that is, the validity of the third assertion above) should be always true, hence the only assumption in this subsection should be the finiteness of the Gibbs measure.

Also note that, at least since Margulis's thesis [Marg], the mixing hypothesis is standard in obtaining precise counting results (see [EM] and the surveys [Bab3, Oh], amongst many references).

In Chapter 10, to prove Theorem 10.4, we will use the following consequence of the mixing property of the Gibbs measures. The forthcoming result of equidistribution of pieces of strong unstable leaves pushed by the geodesic flow, that we state without proof, is due to [Rob1, Coro. 3.2] when $F = 0$ and to [Bab2, Theo. 3] under a more general quasi-product hypothesis, satisfied by our Gibbs measures (see Subsection 3.9).

Theorem 8.2 (Babillot) *Let $\widetilde{M}$ be a complete simply connected Riemannian manifold, with dimension at least 2 and pinched negative curvature at most 1. Let Γ be a nonelementary discrete group of isometries of $\widetilde{M}$. Let $\widetilde{F} : \widetilde{M} \to \mathbb{R}$ be a Γ-invariant Hölder-continuous map. Assume that $\delta = \delta_{\Gamma,\,F}$ is finite and positive, and that the Gibbs measure m_F on T^1M is finite and mixing under the geodesic flow.*

Then for every $v \in T^1\widetilde{M}$, for every relatively compact Borel subset B of $W^{su}(v)$, for every $\psi \in \mathscr{C}_c(T^1M;\mathbb{R})$, if $\widetilde{\psi} = \psi \circ p$ where $p : T^1\widetilde{M} \to T^1M$ is the canonical projection, we have

$$\lim_{t\to+\infty} \int_B \widetilde{\psi} \circ \phi_t(w)\; e^{c_{\widetilde{F}}(w,v)}\; d\mu_{W^{su}(v)}(w) = \int_B e^{c_{\widetilde{F}}(w,v)}\; d\mu_{W^{su}(v)}(w)\; \frac{1}{\|m_F\|} \int_{T^1M} \psi\; dm_F\;.$$

As usual, we may replace continuous functions with compact support by relatively compact Borel subsets with negligible boundary: Under the hypotheses of the above theorem, with v and B as above, for every relatively compact Borel subset A of $T^1\widetilde{M}$ whose boundary has measure 0 for $\widetilde{m}_F$, we have

$$\lim_{t\to+\infty} \sum_{\gamma\in\Gamma} \int_{B\cap\phi_{-t}\gamma A} e^{c_{\widetilde{F}}(w,v)}\; d\mu_{W^{su}(v)}(w) = \int_B e^{c_{\widetilde{F}}(w,v)}\; d\mu_{W^{su}(v)}(w)\; \frac{\widetilde{m}_F(A)}{\|m_F\|}\;.$$

The next subsection gives a finiteness criterion for the Gibbs measures, hence allowing us to use Babillot's theorems 8.1 and 8.2.

8.2 A finiteness criterion for Gibbs states

A point $p \in \partial_\infty\widetilde{M}$ is a *bounded parabolic point* of Γ if it is the fixed point of a parabolic element of Γ and if its stabiliser Γ_p in Γ acts properly with compact quotient on $\Lambda\Gamma - \{p\}$. The discrete nonelementary group of isometries Γ of $\widetilde{M}$ is said to be *geometrically finite* if every element of $\Lambda\Gamma$ is either a conical limit point or a bounded parabolic fixed point of Γ (see for instance [Bowd]).

The following finiteness criterion for Gibbs measures is due to [DaOP] when $F = 0$. With the multiplicative approach explained at the beginning of Chapter 3, it is due to Coudene [Cou2].

Theorem 8.3 *Assume that $\delta_{\Gamma,\,F}$ is finite, and that Γ is geometrically finite with (Γ, F) of divergence type. Then the Gibbs measure m_F is finite if and only if for every parabolic fixed point p of Γ, the series*

$$\sum_{\alpha\in\Gamma_p} d(x,\gamma y)\; e^{\int_x^{\alpha y}(\widetilde{F}-\delta_{\Gamma,\,F})}$$

converges, where Γ_p is the stabiliser of p in Γ.

Proof. We will follow the scheme of proof of [DaOP, Theo. B]. Note that the convergence of the above series depends neither on x nor on y, and that the result is immediate if Γ has no parabolic element, since then the support of the Gibbs measure, which is $\Omega\Gamma$, is compact. Let $\delta = \delta_{\Gamma,\,F}$.

Let Par_Γ be the set of parabolic fixed points of Γ. Since Γ is geometrically finite (see for instance [Bowd]), the action of Γ on Par_Γ has only finitely many orbits, and there exists

a Γ-equivariant family $(\mathscr{H}_p)_{p\in \mathrm{Par}_\Gamma}$ of pairwise disjoint closed horoballs, with $\mathscr{H}_p$ centered at p, such that the quotient

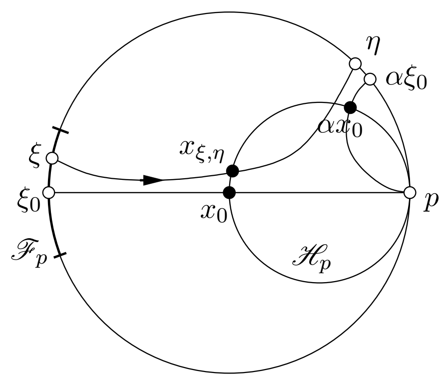
$$M_0 = \Gamma\backslash\big(\mathscr{C}\Lambda\Gamma - \bigcup_{p\in\mathrm{Par}_\Gamma} \mathscr{H}_p\big)$$

is compact. For every $p \in \mathrm{Par}_\Gamma$, we denote by Γ_p the stabiliser of p in Γ, and by $\mathscr{F}_p$ a relatively compact measurable strict fundamental domain for the action of Γ_p on $\Lambda\Gamma - \{p\}$. The inclusion $\mathscr{H}_p \subset \widetilde{M}$ induces an injection $(\Gamma_p\backslash\mathscr{H}_p) \to (\Gamma\backslash\widetilde{M})$ and we will identify $\Gamma_p\backslash\mathscr{H}_p$ with its image. Recall that $\pi : T^1M \to M$ is the canonical projection.

Since the measure m_F on T^1M is finite if and only if its pushforward measure $\pi_* m_F$ on M is finite, since the support of π_*m_F is contained in $\Gamma\backslash\mathscr{C}\Lambda\Gamma$, since M_0 is compact and since there are only finitely many orbits of parabolic fixed points, the Gibbs measure m_F is finite if and only if for every parabolic fixed point p of Γ, we have

$$\pi_* m_F(\Gamma_p\backslash\mathscr{H}_p) < +\infty\ .$$

Fix such a point p. For every $(\xi, \eta) \in \partial^2_\infty \widetilde{M}$, we will denote by $(\xi\eta)$ the geodesic line with endpoints ξ and η oriented from ξ to η; if $\xi \neq p$ and $(\xi\eta)$ meets $\mathscr{H}_p$, we denote by $x_{\xi,\,\eta}$ the first intersection point of $(\xi\eta)$ with $\partial\mathscr{H}_p$. Note that since $\widetilde{M}$ is CAT(-1), if $\xi \neq p$ and if the geodesic lines $(\xi\eta)$ and $(\xi\eta')$ both meet $\mathscr{H}_p$, then $d(x_{\xi,\,\eta}, x_{\xi,\,\eta'}) \leq 2\log(1+\sqrt{2})$ (see for instance [PaP1, Lemma 2.9]).



We define $x_0 = x_{\xi_0,\,p}$ for any fixed $\xi_0 \in \mathscr{F}_p$. In particular, since $\mathscr{F}_p$ is relatively compact in $\partial_\infty\widetilde{M} - \{p\}$, there exists $\kappa \geq 0$ such that for every $\xi \in \mathscr{F}_p$, for every $\eta \in \partial_\infty\widetilde{M} - \{\xi\}$ such that $(\xi\eta)$ meets $\mathscr{H}_p$, we have

$$d(x_0, x_{\xi,\,\eta}) \leq \kappa\ . \tag{98}$$

By the disjointness of the horoballs in the family $(\mathscr{H}_p)_{p\in\mathrm{Par}_\Gamma}$, a geodesic in the support of π_*m_F meeting $\Gamma_p\backslash\mathscr{H}_p$ has a unique lift in $\widetilde{M}$ meeting $\mathscr{H}_p$ starting from the fundamental domain $\mathscr{F}_p$, unless it has a lift which starts from p; its other endpoint is in $\alpha\mathscr{F}_p$ for some $\alpha \in \Gamma_p$, unless it is equal to p. Since Γ is of divergence type, the Patterson densities μ_{x_0} and $\mu^\iota_{x_0}$ have no atom at a non conical limit point (see Corollary 5.12) and the diagonal of $\partial_\infty\widetilde{M} \times \partial_\infty\widetilde{M}$ has zero measure with respect to $\mu^\iota_{x_0} \otimes \mu_{x_0}$ (see Assertion (c) of Proposition 5.5). In particular, $\mu_{x_0}(\mathscr{F}_p)$ and $\mu^\iota_{x_0}(\mathscr{F}_p)$ are positive.

Hence, by the very definition of the Gibbs measure $\widetilde{m}_F$, we have

$$\pi_* m_F(\Gamma_p\backslash\mathscr{H}_p) = \sum_{\alpha\in\Gamma_p} \int_{(\xi,\,\eta)\in\mathscr{F}_p\times\alpha\mathscr{F}_p} \operatorname{length}\big((\xi\eta)\cap\mathscr{H}_p\big)\ \frac{d\mu^\iota_{x_0}(\xi)\,d\mu_{x_0}(\eta)}{D_{F-\delta,\,x_0}(\xi,\eta)^2}\ .$$

By Equation (25), by Equation (98) and by Lemma 3.4 (1), there exists $c>0$ (which depends only on κ, the constants in Lemma 3.4 and $\max_{\pi^{-1}(B(x_0,\,\kappa))}|\widetilde{F}|$) such that, for every $(\xi,\eta)\in\mathscr{F}_p\times\partial_\infty\widetilde{M}$ such that $\xi\neq\eta$ and $(\xi\eta)$ meets $\mathscr{H}_p$, we have

$$\frac{1}{c}\leq\frac{1}{D_{F-\delta,\,x_0}(\xi,\eta)^2}=e^{C_{F\circ\iota-\delta,\,\xi}(x_0,\,x_{\xi,\,\eta})+C_{F-\delta,\,\eta}(x_0,\,x_{\xi,\,\eta})}\leq c\ .$$

Let $(\xi,\eta)\in\mathscr{F}_p\times\alpha\mathscr{F}_p$ be such that $\xi\neq\eta$ and $(\xi\eta)$ meets $\mathscr{H}_p$. The geodesic line from $\alpha^{-1}\eta\in\mathscr{F}_p$ to $\alpha^{-1}\xi$ also meets $\mathscr{H}_p$. The exiting point $y_{\xi,\,\eta}$ of $(\xi\eta)$ out of $\mathscr{H}_p$ is equal to $\alpha\, x_{\alpha^{-1}\eta,\,\alpha^{-1}\xi}$. Hence $d(\alpha x_0,y_{\xi,\,\eta})=d(x_0,x_{\alpha^{-1}\eta,\,\alpha^{-1}\xi})\leq\kappa$. Therefore, by the triangle inequality (see the picture above), we have

$$d(x_0,\alpha x_0)-2\kappa\leq\operatorname{length}\big((\xi\eta)\cap\mathscr{H}_p\big)\leq d(x_0,\alpha x_0)+2\kappa\ .$$

Hence

$$\begin{aligned}\frac{1}{c}\ \mu^\iota_{x_0}(\mathscr{F}_p)\sum_{\alpha\in\Gamma_p}&\big(d(x_0,\alpha x_0)-2\kappa\big)\mu_{x_0}(\alpha\mathscr{F}_p)\\ &\leq\pi_*m_F(\Gamma_p\backslash\mathscr{H}_p)\leq c\ \mu^\iota_{x_0}(\mathscr{F}_p)\sum_{\alpha\in\Gamma_p}\big(d(x_0,\alpha x_0)+2\kappa\big)\mu_{x_0}(\alpha\mathscr{F}_p)\ .\end{aligned}\tag{99}$$

By the equations (34) and (35), for every $\alpha\in\Gamma_p$, we have

$$\mu_x(\alpha\mathscr{F}_p)=\mu_{\alpha^{-1}x_0}(\mathscr{F}_p)=\int_{\xi\in\mathscr{F}_p}e^{-C_{F-\delta,\,\xi}(\alpha^{-1}x_0,\,x_0)}\ d\mu_{x_0}(\xi)\ .\tag{100}$$

When α goes to infinity in the discrete group Γ_p, the point $\alpha^{-1}x_0$ converges to p. Hence, for every $\xi\in\mathscr{F}_p$, except for finitely many α, the set $\mathscr{F}_p$ is contained in the shadow seen from $\alpha^{-1}x_0$ of $B(x_0,\kappa+1)$. Therefore, by Lemma 3.4 (2), there exists a constant $c'>0$ such that for all $\xi\in\mathscr{F}_p$ and $\alpha\in\Gamma_p$, we have

$$\Big|\ C_{F-\delta,\,\xi}(\alpha^{-1}x_0,x_0)+\int_{\alpha^{-1}x_0}^{x_0}(\widetilde{F}-\delta)\ \Big|\leq c'\ .\tag{101}$$

Hence by Equation (100) and by invariance, we have

$$e^{-c'}\mu_{x_0}(\mathscr{F}_p)\ e^{\int_{x_0}^{\alpha x_0}(\widetilde{F}-\delta)}\leq\mu_{x_0}(\alpha\mathscr{F}_p)\leq e^{c'}\mu_{x_0}(\mathscr{F}_p)\ e^{\int_{x_0}^{\alpha x_0}(\widetilde{F}-\delta)}\ .\tag{102}$$

Putting together the equations (99) and (102), there exists $C>0$ such that

$$\begin{aligned}\frac{1}{C}\sum_{\alpha\in\Gamma_p}\big(d(x_0,\alpha x_0)-C\big)e^{\int_{x_0}^{\alpha x_0}(\widetilde{F}-\delta)}&\leq\pi_*m_F(\Gamma_p\backslash\mathscr{H}_p)\\ &\leq C\sum_{\alpha\in\Gamma_p}\big(d(x_0,\alpha x_0)+C\big)e^{\int_{x_0}^{\alpha x_0}(\widetilde{F}-\delta)}\ .\end{aligned}$$

This proves the result. □

A criterion for (Γ,F) to be of divergence type, when Γ is geometrically finite, is the following one, again due to [DaOP] when $F=0$.

Theorem 8.4 *Assume that $\delta_{\Gamma,F}$ is finite and that Γ is geometrically finite. If for every parabolic fixed point p of Γ with stabiliser Γ_p, we have $\delta_{\Gamma_p,F} < \delta_{\Gamma,F}$, then (Γ, F) is of divergence type.*

Proof. Let $(\mu_x)_{x\in\widetilde{M}}$ be the Patterson density for (Γ, F) of dimension $\delta_{\Gamma,F}$ constructed in Proposition 3.9: we use the notation $h, s_k, \overline{Q}_{z,y}(s), \mu_{z,s}$ for $k \in \mathbb{N}, s > \delta_{\Gamma,F}$, $z \in \widetilde{M}$ introduced for that purpose. We start with an independent lemma.

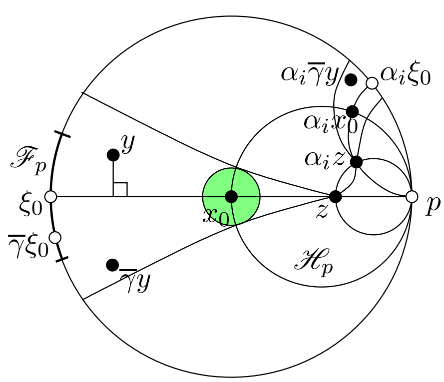
Lemma 8.5 *Let p be a bounded parabolic fixed point of Γ such that $\delta_{\Gamma_p,F} < \delta_{\Gamma,F}$. Then $\mu_x(\{p\}) = 0$.*

Proof. Let $\mathscr{H}_p, \mathscr{F}_p, \kappa, \xi_0, x_0$ be as in the above proof of Theorem 8.3. We may assume that $\xi_0 \notin \Gamma p$. Up to replacing $\mathscr{H}_p$ by a smaller horoball centered at p, we may assume that the closest point to y on the geodesic line $]\xi_0, p[$ between ξ_0 and p does not belong to the geodesic ray $[x_0, p[$. In particular, by the convexity of the horoballs and since $\gamma\mathscr{H}_p \cap \mathscr{H}_p$ is empty if $\gamma \notin \Gamma_p$, the orbit Γy does not meet $\mathscr{H}_p$.

For every $\gamma \in \Gamma$, choose a representative $\overline{\gamma}$ of the left coset of γ in $\Gamma_p\backslash\Gamma$ such that $\overline{\gamma}\xi_0 \in \mathscr{F}_p$ (which is possible since $\gamma\xi_0 \in \Lambda\Gamma - \{p\}$ and $\mathscr{F}_p$ is a fundamental domain for the action of Γ_p on $\Lambda\Gamma - \{p\}$).

Let $\epsilon > 0$ such that $\delta_{\Gamma,F} > \delta_{\Gamma_p,F} + \epsilon$. Let $r_\epsilon \geq 0$ such that $h(t + r) \leq e^{\epsilon t}h(r)$ for all $t \geq 0$ and $r \geq r_\epsilon$. There exists $\kappa' \geq \kappa$ big enough so that if z is the point of $[x_0, p[$ at distance $\max\{r_\epsilon, \kappa' + 2\}$ of x_0, then for every $\gamma \in \Gamma$, there exists a representative $\overline{\gamma}$ of the left coset of γ in $\Gamma_p\backslash\Gamma$ such that $\overline{\gamma}\, y \in \mathscr{O}_z B(x_0, \kappa')$. In particular, $d(z, \Gamma y) \geq r_\epsilon$. Let $\epsilon' \in [0, 1]$ be such that

$$\mu_z\big(\partial(\mathscr{O}_z B(x_0, \kappa' + \epsilon'))\big) = 0\ .$$



Let X be the compact subset $\Gamma y \cup \Lambda\Gamma$ of $\widetilde{M} \cup \partial_\infty \widetilde{M}$, which contains the support of the measures $\mu_{z,s}$ and μ_z for all $z \in \widetilde{M}$ and $s > \delta_{\Gamma,F}$. Let

$$\mathscr{C} = X \cap \bigcup_{\xi\in\mathscr{O}_z B(x_0,\,\kappa'+\epsilon')} [z, \xi]$$

be the set of points of X on a geodesic ray, including its point at infinity, from z through the closed ball $B(x_0, \kappa' + \epsilon')$.

By Equation (98) and by convexity, the set $\mathscr{F}_p$ is contained in the shadow $\mathscr{O}_z B(x_0, \kappa)$, hence in $\mathscr{C}$. Since the point $\overline{\gamma}y$ belongs to $\mathscr{C}$ for every $\gamma \in \Gamma$, we have $X = \{p\} \cup \bigcup_{\alpha\in\Gamma_p} \alpha\mathscr{C}$.

Let $(\alpha_i)_{i\in\mathbb{N}}$ be an enumeration of Γ_p, and for every $n \in \mathbb{N}$, let $U_n = X - \bigcup_{0\leq i\leq n} \alpha_i \mathscr{C}$. Then $(U_n)_{n\in\mathbb{N}}$ is a nonincreasing sequence of neighbourhoods of p in X, with intersection $\{p\}$, and $\mu_z(\partial U_n) = 0$ by the choice of ϵ'. Hence

$$\mu_z(\{p\}) = \lim_{n\to+\infty} \mu_z(U_n)\ .$$

By the convexity of horoballs and a standard argument of quasi-geodesics, there exists a constant $c > 0$ such that for i large enough and for every $\gamma \in \Gamma$, the point $\alpha_i z$ is at distance at most c from the geodesic segment $[z, \alpha_i\overline{\gamma}y]$ (see the picture above). Hence, by (two applications of) Lemma 3.2, there exists a constant $c' > 0$ such that, for every large enough i, for every $\gamma \in \Gamma$, and for every $s \in [\delta_{\Gamma,\,F}, \delta_{\Gamma,\,F} + 1]$, we have

$$\Big|\int_z^{\alpha_i\overline{\gamma}y} (\widetilde{F} - s) - \int_z^{\alpha_i z} (\widetilde{F} - s)\ - \int_z^{\overline{\gamma}y} (\widetilde{F} - s)\Big| \leq c'\ .$$

Since h is nondecreasing, by the triangle inequality and since $d(z, \Gamma y) \geq r_\epsilon$, we have

$$h(d(z, \alpha_i\overline{\gamma}y)) \leq h(d(z, \alpha_i z) + d(z, \overline{\gamma}y)) \leq e^{\epsilon\, d(z,\,\alpha_i z)}\ h(d(z, \overline{\gamma}y))\ .$$

For every $s \in\,]\delta_{\Gamma,\,F}, \delta_{\Gamma,\,F} + 1]$, we hence have, for every n large enough,

$$\begin{aligned}
\mu_{z,\,s}(U_n) &\leq \sum_{i=n+1}^{+\infty}\ \sum_{\overline{\gamma}\in\,\Gamma_p\backslash\Gamma} \frac{1}{\overline{Q}_{y,\,y}(s)}\ h(d(z, \alpha_i\overline{\gamma}y))\ e^{\int_z^{\alpha_i\overline{\gamma}y}(\widetilde{F}-s)}\\
&\leq e^{c'} \sum_{i=n+1}^{+\infty} e^{\int_z^{\alpha_i z}(\widetilde{F}-(s-\epsilon))} \sum_{\overline{\gamma}\in\,\Gamma_p\backslash\Gamma} \frac{1}{\overline{Q}_{y,\,y}(s)}\ h(d(z, \overline{\gamma}y))\ e^{\int_z^{\overline{\gamma}y}(\widetilde{F}-s)}\\
&\leq e^{c'}\ \|\mu_{z,s}\| \sum_{i=n+1}^{+\infty} e^{\int_z^{\alpha_i z}(\widetilde{F}-(s-\epsilon))}\ .
\end{aligned}$$

Taking $s = s_k$ for k large enough, and letting k tends to $+\infty$, we get (note that the constant function 1 on $\partial_\infty\widetilde{M}$ is continuous with compact support and that $\mu_z(\partial U_n) = 0$ for every $n \in \mathbb{N}$)

$$\mu_z(U_n) \leq e^{c'}\ \|\mu_z\| \sum_{i=n+1}^{+\infty} e^{\int_z^{\alpha_i z}(\widetilde{F}-(\delta_{\Gamma,\,F}-\epsilon))}\ .$$

Since $\delta_{\Gamma,\,F} - \epsilon > \delta_{\Gamma_p,\,F}$ and since the remainder of a convergent series tends to 0, we have

$$\mu_z(\{p\}) = \lim_{n\to+\infty} \mu_z(U_n) = 0\ .$$

Since μ_x and μ_z are absolutely continuous one with respect to the other, Lemma 8.5 follows. □

Now, since the limit points of Γ that are not conical ones are (countably many) bounded parabolic fixed points if Γ is geometrically finite, this lemma implies that $\mu_x(\Lambda_c\Gamma) = \mu_x(\Lambda\Gamma) > 0$. Theorem 8.4 hence follows from Corollary 5.10. □

Corollary 8.6 *Assume that $\delta_{\Gamma,\,F}$ is finite, that Γ is geometrically finite, and that $\delta_{\Gamma_p,\,F} < \delta_{\Gamma,\,F}$ for every parabolic fixed point p, where Γ_p is the stabiliser of p in Γ. Then the Gibbs measure m_F is finite.*

In particular, if Γ is geometrically finite, if $\delta_{\Gamma_p} < \delta_\Gamma$ for every $p \in \mathrm{Par}_\Gamma$ (which is in particular the case when $\widetilde{M}$ is a symmetric space), and if $\|F\|_\infty$ is small enough, then m_F is finite, by Lemma 3.3 (iv).

Proof. This is immediate by the theorems 8.4 and 8.3. $\square$

As in [Cou2], this allows us to construct finite (hence ergodic) Gibbs measures for any geometrically finite group Γ, by choosing an appropriate potential F in the preimage in T^1M of the cuspidal parts of M.

9 Growth and equidistribution of orbits and periods

We will give in this chapter precise asymptotic results as t goes to $+\infty$ for the counting functions defined in Subsection 4.1, as corollaries of convergence results for appropriates measures and equidistribution results.

Let $(\widetilde{M}, \Gamma, F)$ be as in the beginning of Chapter 2: $\widetilde{M}$ is a complete simply connected Riemannian manifold, with dimension at least 2 and pinched sectional curvature at most -1; Γ is a nonelementary discrete group of isometries of $\widetilde{M}$; and $\widetilde{F} : T^1\widetilde{M} \to \mathbb{R}$ is a Hölder-continuous Γ-invariant map. Fix x, y in $\widetilde{M}$. Assume that $\delta_{\Gamma,\,F} < +\infty$. Let $\widetilde{m}_F$ be the Gibbs measure on $T^1\widetilde{M}$ associated with a pair of Patterson densities $(\mu^\iota_x)_{x\in\widetilde{M}}$ and $(\mu_x)_{x\in\widetilde{M}}$ for respectively $(\Gamma, F \circ \iota)$ and (Γ, F) of (common) dimension $\delta_{\Gamma,\,F\circ\iota} = \delta_{\Gamma,\,F}$. Denote by $\|m_F\|$ the total mass of the measure m_F on $T^1M = \Gamma\backslash T^1\widetilde{M}$ induced by $\widetilde{m}_F$.

9.1 Convergence of measures on the square product

The aim of this subsection is to prove the following weak star convergence result, whose consequences will be discussed in the next subsections.

Theorem 9.1 *Assume that the critical exponent $\delta_{\Gamma,\,F}$ of (Γ, F) is finite and positive, and that the Gibbs measure m_F is finite and mixing under the geodesic flow on T^1M. As t goes to $+\infty$, the measures*

$$\nu_{x,\,y,\,F,\,t} = \delta_{\Gamma,\,F}\;\|m_F\|\;e^{-\delta_{\Gamma,\,F}\;t}\sum_{\gamma\in\Gamma\,:\,d(x,\gamma y)\le t} e^{\int_x^{\gamma y}\widetilde{F}}\;\mathscr{D}_{\gamma^{-1}x}\otimes\mathscr{D}_{\gamma y}$$

converge to the product measure $\mu^\iota_y \otimes \mu_x$ with respect to the weak star convergence of measures on $(\widetilde{M} \cup \partial_\infty\widetilde{M})^2$.

We will follow closely the proof for the case $F = 0$ given in [Rob1, Chap. 4]. It relies on the next technical proposition, improved in Proposition 9.3. To simplify the notation, we set $\nu_T = \nu_{x,\,y,\,F,\,T}$ for every $T \ge 0$. We recall the notation $\mathscr{C}^\pm_r(z, Z)$, $K^\pm(z, r, Z)$, $K(z, r)$, $v_{\xi,\,\eta,\,z}$, $L_r(z, w)$ and $\mathscr{O}^\pm_r(z, w)$ of Subsection 5.1, as well as the notation of the antipodal map $\iota_z : \partial_\infty\widetilde{M} \to \partial_\infty\widetilde{M}$ with respect to any $z \in \widetilde{M}$ defined in Subsection 2.3.

Proposition 9.2 *Assume that $\delta_{\Gamma,\,F}$ is finite and positive, and that m_F is finite and mixing under the geodesic flow on T^1M. For every $\epsilon > 0$, for all ξ_0 and η_0 in $\partial_\infty\widetilde{M}$ such that $\iota_x(\xi_0) \in \Lambda\Gamma$ and $\iota_y(\eta_0) \in \Lambda\Gamma$, there exist open neighbourhoods V and W of respectively ξ_0 and η_0 in $\partial_\infty\widetilde{M}$ such that for all Borel subsets $A \subset V$ and $B \subset W$,*

$$\limsup_{T\to+\infty}\;\nu_T\big(\mathscr{C}^-_1(x, A) \times \mathscr{C}^-_1(y, B)\big)\;\le\;e^\epsilon\,\mu_x(A)\,\mu^\iota_y(B)$$

$$\text{and} \quad \liminf_{T\to+\infty} \ \nu_T\big(\mathscr{C}_1^+(x,A)\times\mathscr{C}_1^+(y,B)\big) \ \geq \ e^{-\epsilon}\,\mu_x(A)\,\mu^\iota_y(B)\ .$$

Proof. To simplify the notation, let $\delta=\delta_{\Gamma,\,F}>0$. We will denote by $\epsilon_1,\epsilon_2,\epsilon_3,\ldots$ positive functions of ϵ (depending only on δ, on $\max_{\pi^{-1}(B(z,2))}|\widetilde{F}|$ for $z=x,y$, and on the constants $c_1,c_2,c_3,c_4>0$ appearing in Lemma 3.4 for $\widetilde{F}$ and $\widetilde{F}\circ\iota$), that converge to 0 as ϵ goes to 0, and whose exact computation is, though possible, unnecessary.

Let ϵ,ξ_0,η_0 be as in the statement of Proposition 9.2. We first define the neighbourhoods V and W required in the statement of Proposition 9.2.

Let r be in $]0,\min\{1,\epsilon\}[$ possibly outside a countable subset. By the finiteness of the Patterson densities, we may assume that

$$\mu^\iota_x(\partial\mathscr{O}_{\xi_0}B(x,r))=\mu_y(\partial\mathscr{O}_{\eta_0}B(y,r))=0\ . \tag{103}$$

Since the support of the Patterson densities is $\Lambda\Gamma$ (since m_F is finite, see Corollary 5.12 and 5.14), and since $\iota_x(\xi_0)$ and $\iota_y(\eta_0)$ belong to $\Lambda\Gamma$, we have

$$C_r=\mu^\iota_x(\mathscr{O}_{\xi_0}B(x,r))\;\mu_y(\mathscr{O}_{\eta_0}B(y,r))>0\ . \tag{104}$$

By Equation (103) and the convergence property of $\mathscr{O}_r^{\pm}(z,r)$ to $\mathscr{O}_{\xi_0}B(x,r)$ as $z\to\xi_0$ described in Subsection 5.1, there exist open neighbourhoods $\widehat{V},\widehat{W}$ of (respectively) ξ_0,η_0 in $\widetilde{M}\cup\partial_\infty\widetilde{M}$ such that for every r as above, for all $z\in\widehat{V}$ and $w\in\widehat{W}$,

$$e^{-\epsilon}\mu^\iota_x(\mathscr{O}_{\xi_0}B(x,r))\leq\mu^\iota_x(\mathscr{O}_r^{\pm}(z,x))\leq e^{\epsilon}\mu^\iota_x(\mathscr{O}_{\xi_0}B(x,r)) \tag{105}$$

and

$$e^{-\epsilon}\mu_y(\mathscr{O}_{\eta_0}B(y,r))\leq\mu_y(\mathscr{O}_r^{\pm}(w,y))\leq e^{\epsilon}\mu_y(\mathscr{O}_{\eta_0}B(y,r))\ . \tag{106}$$

Finally, let V,W be open neighbourhoods of ξ_0,η_0 in $\partial_\infty\widetilde{M}$ whose closures are contained in $\widehat{V},\widehat{W}$ respectively.

For all Borel subsets A in V and B in W, define (see Subsection 5.1)

$$K^+=K^+(x,r,A)\quad\text{and}\quad K^-=K^-(y,r,B)\ .$$

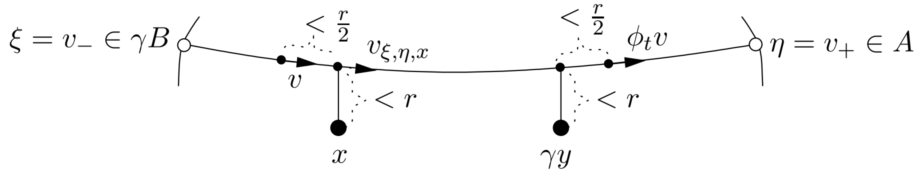
The heart of the proof is to give two pairs of upper and lower bounds (respectively in Equation (108) and Equation (110)) of

$$I_\pm(T)=\int_0^{T\pm3r}e^{\delta\,t}\sum_{\gamma\in\Gamma}\widetilde{m}_F(K^+\cap\phi_{-t}\gamma K^-)\,dt\ .$$

Both of them use the following estimate: For every $(\xi,\eta)\in\partial^2_\infty\widetilde{M}$ such that the orthogonal projection w of x on the geodesic line between ξ and η satisfies $d(w,x)\leq r\leq\min\{1,\epsilon\}$, and in particular if $(\xi,\eta)\in L_r(x,\gamma y)$, we have, by Equation (25) and Lemma 3.4 (1),

$$e^{-\epsilon_1}\leq D_{F-\delta,\,x}(\xi,\eta)=e^{-\frac12(C_{F-\delta,\,\eta}(x,w)+C_{F\circ\iota-\delta,\,\xi}(x,w))}\leq e^{\epsilon_1}\ . \tag{107}$$

First upper and lower bound. We use the definition of the Gibbs measure to estimate individually $\widetilde{m}_F(K^+\cap\phi_{-t}\gamma K^-)$.



For every $\gamma \in \Gamma$ such that $d(x, \gamma y) > 2r$ and for every $t \geq 0$, we have, by the definition of the various geometric sets in Subsection 5.1, that $K^+ \cap \phi_{-t}\gamma K^-$ is the set of $v \in T^1\widetilde{M}$ such that v is r-close to x, $v_+ \in A$, $v_- \in \gamma B$ and $\phi_t v$ is is r-close to γy (see the picture above). Furthermore, by the definition of $\widetilde{m}_F$,

$$\widetilde{m}_F(K^+ \cap \phi_{-t}\gamma K^-) =$$

$$\int_{(\xi,\,\eta)\in L_r(x,\,\gamma y)\cap(\gamma B\times A)} \frac{d\mu^\iota_x(\xi)\; d\mu_y(\eta)}{D_{F-\delta,\,x}(\xi,\eta)^2}\; \int_{-\frac{r}{2}}^{\frac{r}{2}} \mathbb{1}_{K(\gamma y,\,r)}(\phi_{t+s} v_{\xi,\,\eta,\,x})\; ds\;,$$

where $\mathbb{1}_Z$ is the characteristic function of a subset Z.

As in [Rob1, page 59, 60] which only uses an estimate as in Equation (107) and geometric arguments, there exists a constant $c_1 > 0$ such that for every $T > 3r$, if

$$J_\pm(T) = \sum_{\substack{\gamma\,\in\,\Gamma\;:\;d(x,\,\gamma y)\,\leq\,T,\\ \gamma y\,\in\,\mathscr{C}_1^\pm(x,\,A)\,\cap\,\widehat{V},\\ \gamma^{-1}x\,\in\,\mathscr{C}_1^\pm(y,\,B)\,\cap\,\widehat{W}}} \mu^\iota_x\big(\mathscr{O}^\pm_r(\gamma y, x)\big)\; \mu_x\big(\mathscr{O}^\pm_r(x,\gamma y)\big) e^{\delta\, d(x,\gamma y)}\;,$$

then the following two inequalities hold

$$I_-(T) \leq\; e^{\epsilon_2}\; r^2\; J_+(T)\; + c_1 \quad\text{and}\quad I_+(T) \geq\; e^{-\epsilon_2}\; r^2\; J_-(T)\;\; - c_1\;. \tag{108}$$

Second upper and lower bound. Now, we use the mixing property of the geodesic flow to estimate the sum $\sum_{\gamma\in\Gamma} \widetilde{m}_F(K^+ \cap \phi_{-t}\gamma K^-)$.

Since the geodesic flow is mixing, we have, for t large enough,

$$e^{-\epsilon}\; \widetilde{m}_F(K^+)\; \widetilde{m}_F(K^-)\; \leq\; \|m_F\| \sum_{\gamma\in\Gamma} \widetilde{m}_F(K^+ \cap \phi_{-t}\gamma K^-)\; \leq\; e^{\epsilon}\; \widetilde{m}_F(K^+)\; \widetilde{m}_F(K^-)\;. \tag{109}$$

By the definition of $K^+ = K^+(x, r, A)$, we have

$$\widetilde{m}_F(K^+) = r \int_{\eta\in A} d\mu_x(\eta) \int_{\zeta\in\mathscr{O}_\eta B(x,\,r)} D_{F-\delta,\,x}(\zeta,\eta)^{-2}\; d\mu^\iota_x(\zeta)\;.$$

By Equation (107) and by Equation (105) applied to $z \in A$ since $A \subset \widehat{V}$, we have

$$e^{-\epsilon-2\epsilon_1}\; r\; \mu_x(A)\; \mu^\iota_x(\mathscr{O}_{\xi_0} B(x,r))\; \leq \widetilde{m}_F(K^+) \leq\; e^{\epsilon+2\epsilon_1}\; r\; \mu_x(A)\; \mu^\iota_x(\mathscr{O}_{\xi_0} B(x,r))\;.$$

Similarly

$$e^{-\epsilon-2\epsilon_1}\; r\; \mu^\iota_y(B)\; \mu_y(\mathscr{O}_{\eta_0} B(y,r))\; \leq \widetilde{m}_F(K^-) \leq\; e^{\epsilon+2\epsilon_1}\; r\; \mu^\iota_y(B)\; \mu_y(\mathscr{O}_{\eta_0} B(y,r))\;.$$

By taking the product of these inequalities, by multiplying Equation (109) by $e^{\delta t}$ and integrating it over $t \in [0, T \pm 3r]$, we have since $\delta > 0$ and $r \leq \epsilon$, for some constant c_2

independent of T (coming from the fact that the estimations above are valid only for t large enough, hence we have to cut the integral over $t \in [0, T \pm 3r]$ in two), and by the definition of C_r in Equation (104), for T large enough,

$$\delta \, \|m_F\| \; I_-(T) \geq e^{-\epsilon_3} \; r^2 \; C_r \; \mu_x(A) \; \mu^\iota_y(B) \; e^{\delta T} \quad - c_2 \tag{110}$$

$$\text{and} \quad \delta \, \|m_F\| \; I_+(T) \leq e^{\epsilon_3} \; r^2 \; C_r \; \mu_x(A) \; \mu^\iota_y(B) \; e^{\delta T} \quad + c_2 \; .$$

Estimate on C_r. Before showing how these two upper and lower bounds of $I_\pm(T)$ prove the result, let us give a lower and upper estimate on C_r.

By the defining properties of the Patterson densities (see the equations (34) and (35)), we have

$$\mu_y\big(\mathscr{O}^\pm_r(\gamma^{-1}x, y)\big) = \mu_{\gamma y}\big(\mathscr{O}^\pm_r(x, \gamma y)\big) = \int_{\eta \in \mathscr{O}^\pm_r(x,\gamma y)} e^{C_{F-\delta,\,\eta}(x,\gamma y)} \; d\mu_x(\eta) \; .$$

Note that $\mathscr{O}^\pm_r(x, \gamma y) \subset \mathscr{O}_x B(\gamma y, 2r)$ by the convexity of the distance. Hence, by Lemma 3.4 (2) applied to $\widetilde{F} - \delta$ and $y' = \gamma y$, since $\max_{\pi^{-1}(B(\gamma y,\,2r))} |\widetilde{F}| \leq \max_{\pi^{-1}(B(y,\,2))} |\widetilde{F}|$ by invariance, and since $r \leq \epsilon$, we have

$$e^{-\epsilon_4} \; \mu_y\big(\mathscr{O}^\pm_r(\gamma^{-1}x, y)\big) \; \leq \; \mu_x\big(\mathscr{O}^\pm_r(x, \gamma y)\big) \; e^{-\int_x^{\gamma y} (\widetilde{F} - \delta)} \; \leq \; e^{\epsilon_4} \; \mu_y\big(\mathscr{O}^\pm_r(\gamma^{-1}x, y)\big) \; .$$

If $(\gamma y, \gamma^{-1}x) \in \widehat{V} \times \widehat{W}$, we have, respectively by the definition of C_r (see Equation (104)), the definitions of $\widehat{V}$ and $\widehat{W}$ (see the equations (105) and (106)), and the previous inequality,

$$\begin{aligned}
C_r &= \mu^\iota_x(\mathscr{O}_{\xi_0} B(x, r)) \; \mu_y(\mathscr{O}_{\eta_0} B(y, r)) \\
&\geq e^{-2\epsilon} \; \mu^\iota_x(\mathscr{O}^+_r(\gamma y, x)) \; \mu_y(\mathscr{O}^+_r(\gamma^{-1}x, y)) \\
&\geq e^{-2\epsilon - \epsilon_4} \; \mu^\iota_x(\mathscr{O}^+_r(\gamma y, x)) \; \mu_x\big(\mathscr{O}^+_r(x, \gamma y)\big) \; e^{-\int_x^{\gamma y} (\widetilde{F} - \delta)} \; .
\end{aligned}$$

Similarly,

$$C_r \leq e^{2\epsilon + \epsilon_4} \; \mu^\iota_x(\mathscr{O}^-_r(\gamma y, x)) \; \mu_x\big(\mathscr{O}^-_r(x, \gamma y)\big) \; e^{-\int_x^{\gamma y} (\widetilde{F} - \delta)} \; .$$

Conclusion of the proof. Now, respectively by the definition of the measure $\nu_T = \nu_{x,\,y,\,F,\,T}$, by using some constant $c_3 > 0$ independent of T, by the previous lower bound on C_r and the definition of $J_\pm(T)$, by Equation (108), and by Equation (110) with some constant c_4 independent of T, we have

$$\begin{aligned}
C_r \; \nu_T\big(\mathscr{C}^+_1(x, A) \times \mathscr{C}^+_1(y, B)\big) &= \delta \, \|m_F\| \; e^{-\delta T} \sum_{\substack{\gamma \in \Gamma \; : \; d(x, \gamma y) \leq T, \\ (\gamma y, \gamma^{-1} x) \in \mathscr{C}^+_1(x, A) \times \mathscr{C}^+_1(y, B)}} C_r \; e^{\int_x^{\gamma y} \widetilde{F}} \\
&\geq \delta \, \|m_F\| \; e^{-\delta T} \sum_{\substack{\gamma \in \Gamma \; : \; d(x, \gamma y) \leq T, \\ \gamma y \in \mathscr{C}^+_1(x, A) \cap \widehat{V}, \\ \gamma^{-1} x \in \mathscr{C}^+_1(y, B) \cap \widehat{W}}} C_r \; e^{\int_x^{\gamma y} \widetilde{F}} \quad - c_3 \; e^{-\delta T} \\
&\geq \delta \, \|m_F\| \; e^{-\delta T} \; e^{-2\epsilon - \epsilon_4} \; J_+(T) - c_3 \; e^{-\delta T} \\
&\geq \delta \, \|m_F\| \; e^{-\delta T} \; e^{-\epsilon_2 - 2\epsilon - \epsilon_4} \; r^{-2} \big(I_-(T) - c_1\big) - c_3 \; e^{-\delta T} \\
&\geq e^{-\epsilon_5} \; C_r \; \mu_x(A) \; \mu^\iota_y(B) - c_4 \; e^{-\delta T} \; .
\end{aligned}$$

Similarly,

$$C_r \, \nu_T\big(\mathscr{C}_1^-(x,A)\times\mathscr{C}_1^-(y,B)\big) \le e^{\epsilon_5}\, C_r\, \mu_x(A)\, \mu_y^\iota(B) + c_4\, e^{-\delta T}\ .$$

Dividing by $C_r > 0$ (independent of T), the result follows. $\square$

The next result is a small improvement of Proposition 9.2 which gets rid of the assumption that $\iota_x(\xi_0), \iota_y(\eta_0) \in \Lambda\Gamma$, at the price of replacing 1-thickened/1-thinned cones by R-thickened/R-thinned ones, for some $R > 0$ large enough. Note that when $\Lambda\Gamma = \partial_\infty \widetilde{M}$, then this step is not necessary (that is $R = 1$ works). In particular the reader interested only in the case when Γ is cocompact may skip this proposition.

Proposition 9.3 *Assume that $\delta_{\Gamma,F}$ is finite and positive, and that m_F is finite and mixing under the geodesic flow on T^1M. For every $\epsilon' > 0$, for all ξ_0 and η_0 in $\partial_\infty\widetilde{M}$, there exist $R > 0$ and open neighbourhoods V and W of respectively ξ_0 and η_0 in $\partial_\infty\widetilde{M}$ such that for all Borel subsets $A \subset V$ and $B \subset W$,*

$$\limsup_{T\to+\infty}\ \nu_T\big(\mathscr{C}_R^-(x,A)\times\mathscr{C}_R^-(y,B)\big)\ \le\ e^{\epsilon'}\,\mu_x^\iota(A)\,\mu_x(B)$$

$$\text{and}\quad \liminf_{T\to+\infty}\ \nu_T\big(\mathscr{C}_R^+(x,A)\times\mathscr{C}_R^+(y,B)\big)\ \ge\ e^{-\epsilon'}\,\mu_x^\iota(A)\,\mu_x(B)\ .$$

Proof. The idea of the proof is that two cones over the same subset of $\partial_\infty\widetilde{M}$, with vertices two distinct points in $\widetilde{M}$, are asymptotic near infinity, and that we have an appropriate change of basepoint formula for the Patterson measures.

Let $\epsilon > 0$ and $\xi_0, \eta_0 \in \partial_\infty\widetilde{M}$. Fix $\zeta_0 \in \Lambda\Gamma - \{\xi_0, \eta_0\}$ and x_0 (respectively y_0) a point on the geodesic line between ζ_0 and ξ_0 (respectively ζ_0 and η_0). Let

$$R = 1 + \max\{d(x,x_0),\, d(y,y_0)\}$$

and, to simplify the notation, let $\delta = \delta_{\Gamma,F}$.

Let us now define the neighbourhoods V and W required by the statement. Let V_0, W_0 be small enough open neighbourhoods of ξ_0, η_0 respectively in $\partial_\infty\widetilde{M}$, so that Proposition 9.2 holds with x_0, y_0, V_0, W_0 replacing x, y, V, W respectively. Let $\widehat{V_0}, \widehat{W_0}$ be open subsets of $\widetilde{M}\cup\partial_\infty\widetilde{M}$ such that $V_0 = \widehat{V_0}\cap\partial_\infty\widetilde{M}$, $W_0 = \widehat{W_0}\cap\partial_\infty\widetilde{M}$ respectively and such that for all $z \in \widehat{V_0}\cap\widetilde{M}$, $w \in \widehat{W_0}\cap\widetilde{M}$,

$$|\, d(x_0,z) - d(x,z) - \beta_{\xi_0}(x_0,x)\,| \le \epsilon\ , \quad |\, d(y_0,w) - d(y,w) - \beta_{\eta_0}(y_0,y)\,| \le \epsilon\ , \tag{111}$$

$$\Big|\int_x^z \widetilde{F} - \int_{x_0}^z \widetilde{F} + C_{F,\,\xi_0}(x,x_0)\Big| \le \epsilon \quad\text{and}\quad \Big|\int_y^w \widetilde{F}\circ\iota - \int_{y_0}^w \widetilde{F}\circ\iota + C_{F\circ\iota,\,\eta_0}(y,y_0)\Big| \le \epsilon\ . \tag{112}$$

Let us consider neighbourhoods V and W of ξ_0 and η_0, respectively, in $\partial_\infty\widetilde{M}$ whose closures are contained in $\widehat{V_0}$ and $\widehat{W_0}$, respectively.

Now, we start the proof with some computations. If $\gamma y_0, \gamma y \in \widehat{V_0}$ and $\gamma^{-1}x_0, \gamma^{-1}x \in \widehat{W_0}$, the formulae (111) and (112) imply that

$$\begin{aligned} d(x_0,\gamma y_0) \le d(x,\gamma y_0) + \beta_{\xi_0}(x_0,x) + \epsilon &= d(y_0,\gamma^{-1}x) + \beta_{\xi_0}(x_0,x) + \epsilon \\ &\le d(y,\gamma^{-1}x) + \beta_{\eta_0}(y_0,y) + \beta_{\xi_0}(x_0,x) + 2\epsilon \\ &= d(x,\gamma y) - \beta_{\xi_0}(x,x_0) - \beta_{\eta_0}(y,y_0) + 2\epsilon\ , \end{aligned}$$

and that, by Equation (13),

$$e^{\int_x^{\gamma y}\widetilde{F}} \leq e^{\int_{x_0}^{\gamma y}\widetilde{F}-C_{F,\,\xi_0}(x,x_0)+\epsilon} = e^{\int_y^{\gamma^{-1}x_0}\widetilde{F}\circ\iota-C_{F,\,\xi_0}(x,x_0)+\epsilon}$$
$$\leq e^{\int_{y_0}^{\gamma^{-1}x_0}\widetilde{F}\circ\iota-C_{F\circ\iota,\,\eta_0}(y,y_0)-C_{F,\,\xi_0}(x,x_0)+2\epsilon} = e^{\int_{x_0}^{\gamma y_0}\widetilde{F}}\; e^{-C_{F\circ\iota,\,\eta_0}(y,y_0)-C_{F,\,\xi_0}(x,x_0)+2\epsilon}\;.$$

Let $\widehat{V_{-R}} = \{z\in\widetilde{M}\;:\;B(z,R)\subset\widehat{V_0}\}$ and $\widehat{W_{-R}} = \{z\in\widetilde{M}\;:\;B(z,R)\subset\widehat{W_0}\}$, and let A and B be Borel subsets of V and W respectively. Note that if

$$(\gamma y,\gamma^{-1}x)\in\big(\mathscr{C}_R^-(x,A)\times\mathscr{C}_R^-(y,B)\big)\cap\big(\widehat{V_{-R}}\times\widehat{W_{-R}}\big)\;,$$

then

$$(\gamma y_0,\gamma^{-1}x_0)\in\big(\mathscr{C}_1^-(x_0,A)\times\mathscr{C}_1^-(y_0,B)\big)\cap\big(\widehat{V_0}\times\widehat{W_0}\big)$$

by the choice of R and the definition of the r-thickened/r-thinned cones $\mathscr{C}_r^\pm(z,Z)$, and that $\big(\mathscr{C}_R^-(x,A)\times\mathscr{C}_R^-(y,B)\big)\setminus\big(\widehat{V_{-R}}\times\widehat{W_{-R}}\big)$ is a bounded subset of $\widetilde{M}\times\widetilde{M}$.

Now, by the definition of the measure $\nu_T=\nu_{x,\,y,\,F,\,T}$, and by the previous computations, for some constant $c_1'>0$ independent of T, we have

$$\begin{aligned}
&\nu_T\big(\mathscr{C}_R^-(x,A)\times\mathscr{C}_R^-(y,B)\big)\\
&\qquad= \delta\,\|m_F\|\,e^{-\delta T}\sum_{\substack{\gamma\in\Gamma\;:\;d(x,\gamma y)\leq T,\\ (\gamma y,\gamma^{-1}x)\in\mathscr{C}_R^-(x,A)\times\mathscr{C}_R^-(y,B)}} e^{\int_x^{\gamma y}\widetilde{F}}\\
&\qquad\leq \nu_{x_0,\,y_0,\,F,\,T-\beta_{\xi_0}(x,x_0)-\beta_{\eta_0}(y,y_0)+2\epsilon}\big(\mathscr{C}_1^-(x_0,A)\times\mathscr{C}_1^-(y_0,B)\big)\\
&\qquad\quad\times\; e^{-C_{F\circ\iota,\,\eta_0}(y,y_0)-C_{F,\,\xi_0}(x,x_0)+2\epsilon}\;e^{-\delta\beta_{\xi_0}(x,x_0)-\delta\beta_{\eta_0}(y,y_0)+2\epsilon}\;+c_1'e^{-\delta T}\;.
\end{aligned}$$

By Equation (20) and Proposition 9.2, we then have

$$\begin{aligned}
&\limsup_{T\to+\infty}\;\nu_T\big(\mathscr{C}_R^-(x,A)\times\mathscr{C}_R^-(y,B)\big)\\
&\qquad\qquad\leq e^{5\epsilon}\;\mu_{x_0}(A)\;\mu^\iota_{y_0}(B)\;e^{-C_{F\circ\iota-\delta,\,\eta_0}(y,y_0)}\;e^{-C_{F-\delta,\,\xi_0}(x,x_0)}\;.
\end{aligned}$$

By continuity in the equations (111) and (112), since $A\subset V\subset\widehat{V_0}$ and $B\subset W\subset\widehat{W_0}$, we have, for all $\xi\in A$ and $\eta\in B$,

$$|C_{F-\delta,\,\xi}(x,x_0)-C_{F-\delta,\,\xi_0}(x,x_0)|\leq(1+\delta)\epsilon$$

and

$$|C_{F\circ\iota-\delta,\,\eta}(y,y_0)-C_{F\circ\iota-\delta,\,\eta_0}(y,y_0)|\leq(1+\delta)\epsilon\;.$$

By Equation (35) for the Patterson densities $(\mu_x)_{x\in\widetilde{M}}$ and $(\mu^\iota_x)_{x\in\widetilde{M}}$, we hence have

$$\mu_{x_0}(A)\;e^{-C_{F-\delta,\,\xi_0}(x,x_0)}\leq\mu_x(A)\;e^{(1+\delta)\epsilon}$$

and

$$\mu^\iota_{y_0}(B)\;e^{-C_{F\circ\iota-\delta,\,\eta_0}(y,y_0)}\leq\mu^\iota_y(B)\;e^{(1+\delta)\epsilon}\;.$$

Therefore

$$\limsup_{T\to+\infty}\;\nu_T\big(\mathscr{C}_R^-(x,A)\times\mathscr{C}_R^-(y,B)\big)\;\leq\;e^{(7+2\delta)\epsilon}\;\mu_x(A)\;\mu^\iota_y(B)\;.$$

The analogous lower bound is proven similarly, and Proposition 9.3 follows. □

Proof of Theorem 9.1. The end of the proof of Theorem 9.1 follows from Proposition 9.3 exactly as in the "Troisième étape : conclusion" in [Rob1, page 62-63], by replacing μ_x there by μ_x, as well as μ_y by μ^ι_y, $\nu^t_{x,y}$ by $\nu_{x,\,y,\,F,\,t}$, and r by R. □

9.2 Counting orbit points of discrete groups

We deduce in this subsection some corollaries of Theorem 9.1, keeping the notation of the beginning of Chapter 9.

Corollary 9.4 *Assume that the critical exponent $\delta_{\Gamma,\,F}$ of (Γ, F) is finite and positive, and that the Gibbs measure m_F is finite and mixing under the geodesic flow in T^1M. As t goes to $+\infty$, the measures*

$$\delta_{\Gamma,\,F}\;\|m_F\|\;e^{-t\,\delta_{\Gamma,\,F}}\sum_{\gamma\in\Gamma\,:\,d(x,\gamma y)\le t} e^{\int_x^{\gamma y}\widetilde F}\;\mathscr{D}_{\gamma y}$$

converge to the measure $\|\mu^\iota_y\|\;\mu_x$ for the weak star convergence of measures on $\widetilde M\cup\partial_\infty\widetilde M$, and the measures

$$\delta_{\Gamma,\,F}\;\|m_F\|\;e^{-t\,\delta_{\Gamma,\,F}}\sum_{\gamma\in\Gamma\,:\,d(x,\gamma y)\le t} e^{\int_x^{\gamma y}\widetilde F}\;\mathscr{D}_{\gamma^{-1}x}$$

similarly converge to the measure $\|\mu_x\|\;\mu^\iota_y$.

Proof. Since the pushforward of measures by a continuous map is linear and weak star continuous, the result follows from Theorem 9.1 using the projections $(\widetilde M\cup\partial_\infty\widetilde M)^2\to(\widetilde M\cup\partial_\infty\widetilde M)$ on the first factor and on the second factor. □

The second assertion of Corollary 1.4 may also be obtained by exchanging x and y and F and $F\circ\iota$ (see Equation (13) and the last remark of Subsection 5.3).

Corollary 9.5 *Assume that the critical exponent $\delta_{\Gamma,\,F}$ of (Γ, F) is finite and positive, and that the Gibbs measure m_F is finite and mixing under the geodesic flow on T^1M. As t goes to $+\infty$,*

$$\sum_{\gamma\in\Gamma\,:\,d(x,\gamma y)\le t} e^{\int_x^{\gamma y}\widetilde F}\;\sim\;\frac{\|\mu^\iota_y\|\;\|\mu_x\|}{\delta_{\Gamma,\,F}\;\|m_F\|}\;e^{t\,\delta_{\Gamma,\,F}}\;.$$

Proof. This follows by taking the total mass of the above measures on the compact space $\widetilde M\cup\partial_\infty\widetilde M$. □

As another immediate corollary of Corollary 9.4, we obtain the following sharp asymptotic property on the sectorial orbital counting function $G_{\Gamma,\,F,\,x,\,y,\,U}$ (introduced in Subsection 4.1), improving Corollary 4.3 (1).

Corollary 9.6 *Assume that the critical exponent $\delta_{\Gamma,\,F}$ of (Γ, F) is finite and positive, and that the Gibbs measure m_F is finite and mixing under the geodesic flow on T^1M. Let U be an open subset of $\partial_\infty\widetilde M$ meeting $\Lambda\Gamma$ such that $\mu_x(\partial U)=0$. Then as t goes to $+\infty$,*

$$G_{\Gamma,\,F,\,x,\,y,\,U}(t)\;\sim\;\frac{\|\mu^\iota_y\|\;\mu_x(U)}{\delta_{\Gamma,\,F}\;\|m_F\|}\;e^{t\,\delta_{\Gamma,\,F}}\;.$$

Proof. This follows by considering the characteristic functions of cones $\mathscr{C}_xU$ on open subsets U of $\partial_\infty\widetilde M$ (or by taking $V=\partial_\infty\widetilde M$ in the next corollary). □

Similarly, we obtain the following sharp asymptotic property on the bisectorial orbital counting function $G_{\Gamma,\,F,\,x,\,y,\,U,\,V}$ (introduced in Subsection 4.1), improving Corollary 4.3 (2).

Corollary 9.7 *Assume that the critical exponent $\delta_{\Gamma,\,F}$ of (Γ, F) is finite and positive, and that the Gibbs measure m_F is finite and mixing under the geodesic flow on T^1M. Let U and V be two open subsets of $\partial_\infty \widetilde{M}$ meeting $\Lambda\Gamma$ such that $\mu_x(\partial U) = \mu^\iota_x(\partial V) = 0$. Then as t goes to $+\infty$,*

$$G_{\Gamma,\,F,\,x,\,y,\,U,\,V}(t) \;\sim\; \frac{\mu_x(U)\;\mu^\iota_y(V)}{\delta_{\Gamma,\,F}\;\|m_F\|}\;e^{t\,\delta_{\Gamma,\,F}}\;.$$

Proof. Use Theorem 9.1 and consider the product of the characteristic functions of the cones $\mathscr{C}_x U$ and $\mathscr{C}_y V$ on the open subsets U and V of $\partial_\infty \widetilde{M}$. $\Box$

9.3 Equidistribution and counting of periodic orbits of the geodesic flow

The aim of this subsection is to use Theorem 9.1 to prove that the periodic orbits of the geodesic flow in T^1M, appropriately weighted by their periods with respect to the potential, equidistribute with respect to the Gibbs measure. When $F = 0$ and M is convex-cocompact, the result is due to Bowen [Bow1, Bow2, Bow3], see also [Par2]. We follow Roblin's proof when $F = 0$ in [Rob1].

Recall (see Subsection 4.1) that for every $t \in \mathbb{R}$ and for every periodic (not necessarily primitive) orbit g of length $\ell(g)$ of the geodesic flow on T^1M, we denote by $\mathscr{L}_g$ the Lebesgue measure along g, by $\int_g F = \mathscr{L}_g(F)$ the period of g for the potential F, and by $\mathscr{P}\mathrm{er}(t)$ the set (with multiplicities) of periodic orbits of the geodesic flow on T^1M with length at most $t \in \mathbb{R}$.

As in Bowen's two equidistribution results in the convex-cocompact case, the first assertion of the theorem below claims the equidistribution of the Lebesgue measures of the closed orbits when weighted by the exponential of their periods for the potential, the second one claims the equidistribution of their Lebesgue means.

Theorem 9.8 *Assume that the critical exponent $\delta_{\Gamma,\,F}$ of (Γ, F) is finite and positive, and that the Gibbs measure m_F is finite and mixing under the geodesic flow of T^1M.*

(1) As t goes to $+\infty$, the measures

$$\delta_{\Gamma,\,F}\;\; e^{-\delta_{\Gamma,\,F}\;t} \sum_{g\in\mathscr{P}\mathrm{er}(t)} e^{\mathscr{L}_g(F)}\;\mathscr{L}_g$$

converge to the normalised Gibbs measure $\frac{m_F}{\|m_F\|}$ with respect to the weak star convergence of measures on T^1M.

(2) As t goes to $+\infty$, the measures

$$\delta_{\Gamma,\,F}\;t\;\; e^{-\delta_{\Gamma,\,F}\;t} \sum_{g\in\mathscr{P}\mathrm{er}(t)} e^{\mathscr{L}_g(F)}\;\frac{\mathscr{L}_g}{\ell(g)}$$

weak star converge to $\frac{m_F}{\|m_F\|}$.

Proof. Let $\delta = \delta_{\Gamma,\,F}$. Let us denote by $\mathscr{H}_\Gamma$ the set of hyperbolic elements of Γ. For every $\gamma \in \mathscr{H}_\Gamma$, let $\ell(\gamma)$ be its translation length, let Axe_γ be its translation axis, oriented from the repulsive fixed point γ_- to the attractive one γ_+, and let $\mathscr{L}_\gamma$ be the measure on $T^1\widetilde{M}$

which is the lift to $T^1\widetilde{M}$ of the Lebesgue measure along the translation axis of γ: For every map $f : T^1\widetilde{M} \to \mathbb{R}$ which is continuous with compact support, we have

$$\mathscr{L}_\gamma(f) = \int_{t\in\mathbb{R}} f(\phi_t v)\; dt$$

where v is any unit tangent vector to the oriented geodesic line Axe_γ. Recall that the period of γ for F is $\mathrm{Per}_F(\gamma) = \int_0^{\ell(\gamma)} \widetilde{F}(\phi_t v)\; dt$ with v as above. Note that $\alpha_* \mathscr{L}_\gamma = \mathscr{L}_{\alpha\gamma\alpha^{-1}}$ for every $\alpha \in \Gamma$. The set $\mathscr{H}_{\Gamma,\, t}$ of hyperbolic elements of Γ whose translation length is at most t is invariant under conjugation by elements of Γ.

(1) Let us prove the first statement of Theorem 9.8. By the Γ-invariance of the measures, and by the definition of the weak star convergence, we only have to show that, for every compact subset K of $\widetilde{M}$, as t goes to $+\infty$, the measures

$$\delta\|m_F\|\;\; e^{-\delta\, t} \sum_{\gamma\in\mathscr{H}_{\Gamma,\, t}} e^{\mathrm{Per}_F(\gamma)}\; \mathscr{L}_\gamma$$

restricted to the compact subset $\pi^{-1}(K)$ of $T^1\widetilde{M}$ converge to $\widetilde{m}_F$ restricted to $\pi^{-1}(K)$ with respect to the weak star convergence of measures on $\pi^{-1}(K)$.

Fix a compact subset K of M. Let $\epsilon \in\,]0, \frac{1}{2}]$. As in the proof of Proposition 9.2, we will denote by $\epsilon_1, \epsilon_2, \epsilon_3$ positive functions of ϵ (depending only on $\delta > 0$, on $\max_{\pi^{-1}(\mathscr{N}_1(K))} |\widetilde{F}|$, and on the constants $c_1, c_2, c_3, c_4 > 0$ appearing in Lemma 3.4 for $\widetilde{F}$ as well as for $\widetilde{F}\circ\iota$), that converge to 0 as ϵ goes to 0.

For every $x \in K$, let $V(x, \epsilon)$ be the set of pairs (ξ, η) of distinct elements in $\widetilde{M} \cup \partial_\infty \widetilde{M}$ such that the geodesic segment ray or line between ξ and η meets $B(x, \epsilon)$. As in Equation (107), by Equation (25) and Lemma 3.4 (1), for every $(\xi, \eta) \in V(x, \epsilon) \cap \partial^2_\infty \widetilde{M}$, we have

$$e^{-\epsilon_1} \le D_{F-\delta,\, x}(\xi, \eta) \le e^{\epsilon_1}\;. \tag{113}$$

For all $x \in K$ and $t > 0$, let ν and ν_t be the measures on $(\widetilde{M} \cup \partial_\infty \widetilde{M})^2$ defined by :

$$d\nu(\xi, \eta) = \frac{\mu^\iota_x(\xi) \otimes d\mu_x(\eta)}{D_{F-\delta,\, x}(\xi, \eta)^2}\;, \qquad \nu_t = \delta\|m_F\| e^{-\delta t} \sum_{\gamma\in\Gamma,\; d(x,\gamma x)\le t} e^{\int_x^{\gamma x} \widetilde{F}}\; \mathscr{D}_{\gamma^{-1}x} \otimes \mathscr{D}_{\gamma x}\;.$$

By Theorem 9.1 (taking $y = x$), we know that ν_t weak star converges to $\mu^\iota_x \otimes \mu_x$ as $t \to +\infty$. Let ψ be a continuous nonnegative map with support in $V(x, \epsilon)$. By Equation (113), we hence have

$$e^{-\epsilon_1}\; \nu(\psi) \le \liminf_{t\to+\infty} \nu_t(\psi) \le \limsup_{t\to+\infty} \nu_t(\psi) \le e^{\epsilon_1}\; \nu(\psi)\;.$$

Let

$$\nu'_t = \delta\|m_F\| e^{-\delta t} \sum_{\gamma\in\mathscr{H}_\Gamma,\; \ell(\gamma)\le t} e^{\mathrm{Per}_F(\gamma)}\;\; \mathscr{D}_{\gamma_-} \otimes \mathscr{D}_{\gamma_+}\;.$$

Note that by Gromov's hyperbolicity criterion (see also Lemma 2.7), for every $\gamma \in \Gamma$, if $(\gamma^{-1}x, \gamma x) \in V(x, \epsilon)$ and $d(x, \gamma x)$ is large enough, then γ is hyperbolic and x is at distance at most 2ϵ from the translation axis Axe_γ of γ. In particular

$$\ell(\gamma) \le d(x, \gamma x) \le \ell(\gamma) + 4\epsilon\;.$$

Furthermore, $\gamma^{\pm 1}x$ is close to $\gamma_\pm$ uniformly in γ, hence so is $\mathscr{D}_{\gamma^{\pm 1}x}$ to $\mathscr{D}_{\gamma_\pm}$. By Lemma 3.2 and the remark following it, if p is the closest point on Axe_γ to x, we hence have

$$\Big|\int_x^{\gamma x}\widetilde{F}-\operatorname{Per}_F(\gamma)\Big| = \Big|\int_x^{\gamma x}\widetilde{F}-\int_p^{\gamma p}\widetilde{F}\Big|$$
$$\leq 2\,c_3(2\epsilon)^{c_4}+2\,\epsilon\Big(\max_{\pi^{-1}(B(x,\,2\epsilon))}|\widetilde{F}|+\max_{\pi^{-1}(B(\gamma x,\,2\epsilon))}|\widetilde{F}|\Big)\,,$$

which tends to 0 as ϵ goes to 0 uniformly in γ by the Γ-invariance of $\widetilde{F}$. Hence if t is large enough, then

$$e^{-\epsilon_2}\nu_t(\psi)\leq\nu'_t(\psi)\leq e^{\epsilon_2}\nu_t(\psi)\;.$$

On $T^1\widetilde{M}$ identified with $\partial^2_\infty\widetilde{M}\times\mathbb{R}$ by one of the Hopf parametrisations, note that

$$\frac{1}{\|m_F\|}\,\nu'_t\otimes ds=\delta\;e^{-\delta t}\sum_{\gamma\in\mathscr{H}_\Gamma,\;\ell(\gamma)\leq t}e^{\operatorname{Per}_F(\gamma)}\mathscr{L}_\gamma\;,$$

and denote by ν''_t the measure on the right. Therefore, for every continuous map ψ' : $T^1\widetilde{M}\rightarrow\mathbb{R}$ with compact support in $V(x,\epsilon)\times\mathbb{R}$, which is a product of two continuous maps on each of the two variables of this product, we have

$$e^{-\epsilon_3}\;\frac{m_F(\psi')}{\|m_F\|}\leq\liminf_{t\rightarrow+\infty}\nu''_t(\psi')\leq\limsup_{t\rightarrow+\infty}\nu''_t(\psi')\leq e^{\epsilon_3}\;\frac{m_F(\psi')}{\|m_F\|}\;.$$

Approximating uniformly continuous maps with compact support in $V(x,\epsilon)\times\mathbb{R}$ by finite linear combinations of product maps, covering $\pi^{-1}(K)$ by finitely many sets $V(x,\epsilon)\times\mathbb{R}$, using a partition of unity and letting ϵ goes to 0, the first assertion of Theorem 9.8 follows.

(2) The deduction of the second assertion from the first one is standard. Consider the measures

$$m'_t=\delta e^{-\delta t}\sum_{g\in\mathscr{P}\mathrm{er}(t)}e^{\mathscr{L}_g(F)}\;\mathscr{L}_g\quad\text{and}\quad m''_t=\delta te^{-\delta t}\sum_{g\in\mathscr{P}\mathrm{er}(t)}e^{\mathscr{L}_g(F)}\;\frac{\mathscr{L}_g}{\ell(g)}$$

on T^1M. Fix a continuous map $\psi:T^1M\rightarrow[0,+\infty[$ with compact support. For every $\epsilon>0$, for every $t>0$, we have

$$\begin{aligned}m''_t(\psi)\geq m'_t(\psi)&\geq\delta e^{-\delta t}\sum_{g\in\mathscr{P}\mathrm{er}(t)-\mathscr{P}\mathrm{er}(e^{-\epsilon}t)}e^{\mathscr{L}_g(F)}\;\mathscr{L}_g(\psi)\\&\geq e^{-\epsilon}\delta te^{-\delta t}\sum_{g\in\mathscr{P}\mathrm{er}(t)-\mathscr{P}\mathrm{er}(e^{-\epsilon}t)}e^{\mathscr{L}_g(F)}\;\frac{\mathscr{L}_g(\psi)}{\ell(g)}\\&=e^{-\epsilon}m''_t(\psi)-e^{-\epsilon}\delta t\;e^{-\delta t}\sum_{g\in\mathscr{P}\mathrm{er}(e^{-\epsilon}t)}e^{\mathscr{L}_g(F)}\;\frac{\mathscr{L}_g(\psi)}{\ell(g)}\end{aligned}$$

Since the closed orbits meeting the support of ψ have a positive lower bound on their lengths, and by the first assertion of Theorem 9.8, there exists a constant $c>0$ such that the second term of the above difference is at most $c\,e^{-\delta t}e^{\delta\,e^{-\epsilon}t}$, which tends to 0 as t tends to infinity. Hence by applying twice the first assertion of Theorem 9.8, we have

$$\frac{m_F(\psi)}{\|m_F\|}=\lim_{t\rightarrow+\infty}m'_t(\psi)\leq\liminf_{t\rightarrow+\infty}m''_t(\psi)\leq\limsup_{t\rightarrow+\infty}m''_t(\psi)\leq\lim_{t\rightarrow+\infty}e^{\epsilon}m'_t(\psi)=e^{\epsilon}\frac{m_F(\psi)}{\|m_F\|}\;,$$

and the result follows by letting ϵ go to 0 (and writing any continuous map $\psi : T^1M \to \mathbb{R}$ with compact support into the sum of its positive and negative parts). $\square$

Applying this equidistribution result to characteristic functions, for every relatively compact open subset U of T^1M whose boundary has Gibbs measure 0, we have that, as $t \to +\infty$,

$$\sum_{g\in\mathscr{P}\mathrm{er}(t)} e^{\mathrm{Per}_F(g)} \frac{\ell(g\cap U)}{\ell(g)} \;\sim\; \frac{e^{\delta_{\Gamma,\,F}\,t}}{\delta_{\Gamma,\,F}\,t} \frac{m_F(U)}{\|m_F\|}\;.$$

As there exists a compact subset of T^1M containing all closed orbits of the geodesic flow if Γ is convex-cocompact, the following corollary holds.

Corollary 9.9 *If Γ is convex-cocompact, if $(\phi_t)_{t\in\mathbb{R}}$ is topologically mixing, if $\delta_{\Gamma,\,F}$ is finite and positive, then as $t \to +\infty$,*

$$\sum_{g\in\mathscr{P}\mathrm{er}(t)} e^{\mathrm{Per}_F(g)} \sim \frac{e^{\delta_{\Gamma,\,F}\,t}}{\delta_{\Gamma,\,F}\,t}\;.$$

Let $\mathscr{C}_b(T^1M)^*$ be the dual topological space of the Banach space of real bounded continuous functions on T^1M. By the *narrow convergence* of finite measures, we mean as usual the convergence for the weak star topology in $\mathscr{C}_b(T^1M)^*$. This result can be improved, and holds when Γ is geometrically finite, though the extension requires more work. For this, we improve, under the geometrically finiteness condition, Theorem 9.8 from weak star convergence to narrow convergence, as for the case $F = 0$ in [Rob1]. The main point is to prove that there is no loss of mass in the cuspidal parts during the convergence process.

Theorem 9.10 *Assume that the critical exponent $\delta_{\Gamma,\,F}$ of (Γ, F) is finite and positive, that the Gibbs measure m_F is finite and mixing under the geodesic flow on T^1M, and that Γ is geometrically finite*

(1) As t goes to $+\infty$, the measures

$$\delta_{\Gamma,\,F}\;\, e^{-\delta_{\Gamma,\,F}\;t} \sum_{g\in\mathscr{P}\mathrm{er}(t)} e^{\mathscr{L}_g(F)}\;\mathscr{L}_g$$

converge to the normalised Gibbs measure $\frac{m_F}{\|m_F\|}$ with respect to the narrow convergence.

(2) As t goes to $+\infty$,

$$\delta_{\Gamma,\,F}\;t\;\; e^{-\delta_{\Gamma,\,F}\;t} \sum_{g\in\mathscr{P}\mathrm{er}(t)} e^{\mathscr{L}_g(F)}\;\frac{\mathscr{L}_g}{\ell(g)}$$

converges to $\frac{m_F}{\|m_F\|}$ with respect to the narrow convergence.

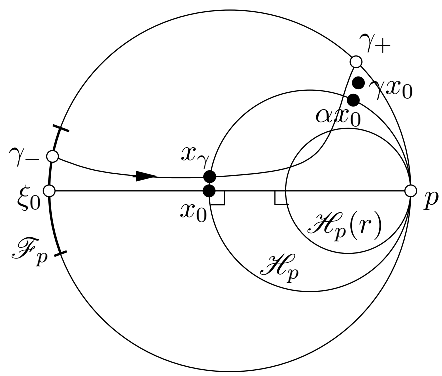
Proof. The deduction of Assertion (2) from Assertion (1) proceeds as in the proof of Theorem 9.8, using the fact that since Γ is geometrically finite, there exists a compact set of M meeting (though not necessarily containing) all closed geodesics, hence there exists $c' > 0$ such that every closed geodesic of M has length at least c'.

Let us prove Assertion (1). Let $\delta = \delta_{\Gamma,\,F}$. For $t \geq 0$, consider the measure

$$m_t = \delta\, e^{-\delta t} \sum_{g \in \mathscr{P}\mathrm{er}(t)} e^{\mathscr{L}_g(F)}\, \mathscr{L}_g$$

on T^1M and let $\pi_* m_t$ be its pushforward on M. We will use the notation of the proof of Theorem 8.3, in particular $\mathrm{Par}_\Gamma, \mathscr{H}_p, \Gamma_p, \mathscr{F}_p, x_0, \kappa$ for every parabolic fixed point p of Γ.

For every $r \geq 0$ and for every parabolic fixed point p of Γ, let $\mathscr{H}_p(r)$ be the horoball centered at p, contained in $\mathscr{H}_p$ such that the distance between the horospheres $\partial\mathscr{H}_p(r)$ and $\partial\mathscr{H}_p$ is r.



By the finiteness of the number of orbits of parabolic fixed points under Γ and by the compactness of the quotient $\Gamma\backslash\big(\mathscr{C}\Lambda\Gamma - \bigcup_{p\in\mathrm{Par}_\Gamma}\mathscr{H}_p(r)\big)$, we only have to prove that for every parabolic fixed point p of Γ,

$$\lim_{r\to+\infty}\ \limsup_{t\to+\infty} \pi_* m_t(\Gamma_p\backslash\mathscr{H}_p(r)) = 0\ .$$

Fix such a point p. A closed geodesic in M meeting $\Gamma_p\backslash\mathscr{H}_p$ has a unique lift in $\widetilde{M}$ meeting $\mathscr{H}_p$ starting from the fundamental domain $\mathscr{F}_p$ for the action of Γ_p on $\Lambda\Gamma - \{p\}$; its other endpoint is in $\alpha\mathscr{F}_p$ for some $\alpha \in \Gamma_p$. For all $t \geq 0$ and $\alpha \in \Gamma_p$, let $\Gamma(t,\alpha)$ be the set of hyperbolic elements γ of Γ with repulsive fixed point γ_- in $\mathscr{F}_p$, attractive fixed point γ_+ in $\alpha\mathscr{F}_p$ and translation length $\ell(\gamma)$ at most t. Then

$$\pi_* m_t(\Gamma_p\backslash\mathscr{H}_p(r)) = \delta e^{-\delta t} \sum_{\alpha\in\Gamma_p}\ \sum_{\gamma\in\Gamma(t,\,\alpha)} e^{\mathrm{Per}_F(\gamma)}\ \mathrm{length}(Axe_\gamma \cap \mathscr{H}_p(r))\ . \tag{114}$$

Let $\gamma \in \Gamma(t,\alpha)$ be such that its translation axis Axe_γ meets $\mathscr{H}_p(r)$ (note that for the other ones, we have $\mathrm{length}(Axe_\gamma \cap \mathscr{H}_p(r)) = 0$). We orient Axe_γ from γ_- to γ_+, and we denote by x_γ the entering point of Axe_γ in $\mathscr{H}_p$. By Equation (98), we have

$$d(x_0, x_\gamma) \leq \kappa\ .$$

Since the distance between any point of $\partial\mathscr{H}_p$ and any point of $\mathscr{H}_p(r)$ is at least r, we have (see the above picture),

$$\mathrm{length}(Axe_\gamma \cap \mathscr{H}_p(r)) \leq d(x_0, \alpha x_0) - 2r + 2\kappa\ . \tag{115}$$

In particular,

$$d(x_0, \alpha x_0) \geq 2r - 2\kappa \; . \tag{116}$$

Consider the constants R and C given by Mohsen's shadow lemma 3.10 applied to $(\mu_x)_{x\in\widetilde{M}}$ as above and $K = \{x_0\}$, and let $\mathscr{O}_\gamma = \mathscr{O}_{x_0} B(\gamma x_0, R)$. The point αx_0 is at distance uniformly bounded from Axe_γ. Hence $x_0 = \alpha^{-1}\alpha x_0$ is at distance uniformly bounded from $[\alpha^{-1}x_0, \alpha^{-1}\gamma x_0]$. If r is large enough, then $d(x_0, \alpha^{-1}x_0)$ is large, hence $\alpha^{-1}x_0$ is close to p. Therefore, there exists a compact subset K (independent of t, r, α, γ) of $\partial_\infty \widetilde{M} - \{p\}$ such that the set $\alpha^{-1}\mathscr{O}_\gamma = \mathscr{O}_{\alpha^{-1}x_0} B(\alpha^{-1}\gamma x_0, R)$ is contained in K for r large enough.

Since $d(x_0, x_\gamma) \leq \kappa$ and by (two applications of) Lemma 3.2, there exists $c_1 \geq 0$ such that

$$\Big| \operatorname{Per}_F(\gamma) - \int_{x_0}^{\gamma x_0} \widetilde{F} \;\Big| = \Big| \int_{x_\gamma}^{\gamma x_\gamma} \widetilde{F} \; - \int_{x_0}^{\gamma x_0} \widetilde{F} \;\Big| \leq c_1 \; .$$

Since x_0 is at distance at most κ from Axe_γ, we have

$$d(x_0, \gamma x_0) \leq \ell(\gamma) + 2\kappa \leq t + 2\kappa \; ,$$

and if furthermore $\gamma \notin \Gamma(t-1, \alpha)$, then

$$t - 1 \leq \ell(\gamma) \leq d(x_0, \gamma x_0) \; .$$

Hence, by Mohsen's shadow lemma 3.10,

$$e^{\operatorname{Per}_F(\gamma)} \leq e^{c_1} e^{\int_{x_0}^{\gamma x_0} \widetilde{F}} \leq e^{c_1 + 2\delta\kappa} e^{\delta t} e^{\int_{x_0}^{\gamma x_0} (\widetilde{F} - \delta)} \leq C\, e^{c_1 + 2\delta\kappa} e^{\delta t} \mu_{x_0}(\mathscr{O}_\gamma) \; .$$

By the discreteness of Γx_0, there exists $N > 0$ (independent of t) such that a point $\xi \in \partial_\infty \widetilde{M}$ belongs to at most N shadows $\mathscr{O}_\gamma$ for $\gamma \in \Gamma$ with $t - 1 \leq d(x_0, \gamma x_0) \leq t + 2\kappa$. Since the shadow $\mathscr{O}_\gamma$ is contained in αK if Axe_γ meets $\mathscr{H}_p(r)$ for r large enough, we hence have, for r large enough,

$$\sum_{\gamma \in \Gamma(t,\alpha) - \Gamma(t-1,\alpha)\,:\; Axe_\gamma \cap \mathscr{H}_p(r) \neq \emptyset} e^{\operatorname{Per}_F(\gamma)} \leq CN\, e^{c_1 + 2\delta\kappa}\, e^{\delta t} \mu_{x_0}(\alpha K) \; .$$

By a summation, there exists therefore a constant $c_2 > 0$ such that

$$\sum_{\gamma \in \Gamma(t,\alpha)\,:\; Axe_\gamma \cap \mathscr{H}_p(r) \neq \emptyset} e^{\operatorname{Per}_F(\gamma)} \leq c_2\, e^{\delta t} \mu_{x_0}(\alpha K) \; . \tag{117}$$

By the equations (34), (35) and (101), we have, for some constant $c_3 > 0$,

$$\mu_{x_0}(\alpha K) = \mu_{\alpha^{-1}x_0}(K) = \int_{\xi \in K} e^{-C_{F-\delta,\,\xi}(\alpha^{-1}x_0,\, x_0)}\, d\mu_{x_0}(\xi) \leq c_3 \mu_{x_0}(K)\, e^{\int_{x_0}^{\alpha x_0} (\widetilde{F} - \delta)} \; . \tag{118}$$

By the equations (114), (115), (116), (117) and (118), for r large enough and for every $t \geq 0$, we hence have

$$\begin{aligned} &\pi_* m_t(\Gamma_p \backslash \mathscr{H}_p(r)) \\ &\leq \delta c_2 c_3 \mu_{x_0}(K) \sum_{\alpha \in \Gamma_p\,:\; d(x_0,\, \alpha x_0) \geq 2r - 2\kappa} (d(x_0, \alpha x_0) - 2r + 2\kappa)\; e^{\int_{x_0}^{\alpha x_0} (\widetilde{F} - \delta)} \; . \end{aligned}$$

The result then follows from Theorem 8.3, since we assume m_F to be finite (which implies that (Γ, F) is of divergence type by Corollary 5.14). □

The next result is immediate, applying the second assertion above to the (bounded) constant function 1. By Subsection 8.2, the finiteness of m_F (which was automatic in the convex-cocompact case but no longer is in the geometrically finite one) follows if $\delta_{\Gamma_p,\,F} < \delta_{\Gamma,\,p}$ for every parabolic fixed point p of Γ.

Corollary 9.11 *Assume that* Γ *is geometrically finite,* $(\phi_t)_{t\in\mathbb{R}}$ *is topologically mixing,* $\delta_{\Gamma,\,F}$ *is finite and positive, and* m_F *is finite. Then as* $t \to +\infty$*, we have*

$$\sum_{g\in\mathscr{P}\mathrm{er}(t)} e^{\mathrm{Per}_F(g)} \sim \frac{e^{\delta_{\Gamma,\,F}\,t}}{\delta_{\Gamma,\,F}\,t}\;.$$

10 The ergodic theory of the strong unstable foliation

Let $(\widetilde{M}, \Gamma, \widetilde{F})$ be as in the beginning of Chapter 2: $\widetilde{M}$ is a complete simply connected Riemannian manifold, with dimension at least 2 and pinched sectional curvature at most -1; Γ is a nonelementary discrete group of isometries of $\widetilde{M}$; and $\widetilde{F} : T^1\widetilde{M} \to \mathbb{R}$ is a Hölder-continuous Γ-invariant map. Let $x_0 \in \widetilde{M}$ and $\delta = \delta_{\Gamma,\,F}$. The aim of this chapter is to prove a result of unique ergodicity for the strong unstable foliation of $T^1M = \Gamma\backslash T^1\widetilde{M}$ endowed with the conditional measures of Gibbs measures. We refer for instance to [Wal, BoM] for generalities on unique ergodicity of measurable dynamical systems and foliations, though in our case we will (and have to, in order to take into account the potential) consider quasi-invariant measures instead of invariant ones.

10.1 Quasi-invariant transverse measures

We first recall some definitions. Let N be a smooth manifold of dimension n endowed with a smooth action of a discrete group G and a C^0 foliation $\mathscr{F}$ of dimension k invariant by G. A *transversal* to $\mathscr{F}$ is a C^0 submanifold T of N such that for every $x \in T$, there exists a foliated chart $\varphi : U \to \mathbb{R}^k \times \mathbb{R}^{n-k}$ at x, sending x to 0 and $T \cap U$ to $\{0\} \times \mathbb{R}^{n-k}$. Let $\mathscr{T}(\mathscr{F})$ be the set of transversals to $\mathscr{F}$. A *holonomy map* for $\mathscr{F}$ is a homeomorphism $f : T \to T'$ between two transversals to $\mathscr{F}$, such that $f(x)$ is in the same leaf of $\mathscr{F}$ as x for every $x \in T$. A *G-invariant cocycle* for the foliation $\mathscr{F}$ is a continuous map c defined on the subspace of $N \times N$ of pairs of points in a same leaf of $\mathscr{F}$, with real values, satisfying $c(u, v) + c(v, w) = c(u, w)$ and $c(u, v) = c(\gamma u, \gamma v)$ for all triples (u, v, w) of points in a same leaf of $\mathscr{F}$ and every $\gamma \in G$. Given a G-invariant cocycle c for $\mathscr{F}$, a *G-equivariant c-quasi-invariant transverse measure for* $\mathscr{F}$ is a family $\nu = (\nu_T)_{T\in\mathscr{T}(\mathscr{F})}$, where ν_T is a (not necessarily finite) locally finite (positive Borel) measure on the transversal T, nonzero for at least one T, satisfying the following properties for all $T, T' \in \mathscr{T}(\mathscr{F})$:

(i) if $T' \subset T$, then $(\nu_T)_{|T'} = \nu_{T'}$,

(ii) $\gamma_*\nu_T = \nu_{\gamma T}$ for every $\gamma \in G$,

(iii) for all holonomy maps $f : T \to T'$, the measures $\nu_{T'}$ and $f_*\nu_T$ on T' are equivalent, and their Radon-Nikodym derivative satisfies, for every $x \in T$,

$$\frac{d\,\nu_{T'}}{d\,f_*\nu_T}(f(x)) = e^{c(f(x),\,x)}\;.$$

When $c = 0$, such a family $\nu = (\nu_T)_{T\in\mathscr{T}(\mathscr{F})}$ is said to be *invariant under holonomy* : for all holonomy maps $f : T \to T'$, we have $f_*\nu_T = \nu_{T'}$. Such a family $\nu = (\nu_T)_{T\in\mathscr{T}(\mathscr{F})}$ is *ergodic* if, given a G-invariant subset A of N, which is a union of leaves of $\mathscr{F}$, whose intersection with any transversal is measurable, then either for all transversals T we have $\nu_T(A \cap T) = 0$, or for all transversals T we have $\nu_T({}^cA \cap T) = 0$.

We will apply these definitions with $N = T^1\widetilde{M}$, $G = \Gamma$, $\mathscr{F}$ the strong unstable foliation $\widetilde{\mathscr{W}}^{su}$ of $T^1\widetilde{M}$ (defined in Subsection 2.4) and with the (Γ-invariant) cocycle $c = c_{\widetilde{F}}$ defined as follows: for all v, w in the same leaf of $\widetilde{\mathscr{W}}^{su}$,

$$c_{\widetilde{F}}(v, w) = \lim_{t\to+\infty} \int_0^t \widetilde{F}(\phi_{-t}v) - \widetilde{F}(\phi_{-t}w)\, dt = -C_{F\circ\iota - \delta,\, v_-}(\pi(v), \pi(w)) \ . \tag{119}$$

Note that when F is constant, this cocycle is equal to 0. We will say that a Γ-equivariant $c_{\widetilde{F}}$-quasi-invariant transverse measure $\nu = (\nu_T)_{T\in\mathscr{T}(\widetilde{\mathscr{W}}^{su})}$ for $\widetilde{\mathscr{W}}^{su}$ *gives full measure to the negatively recurrent set* if for all transversals T to $\widetilde{\mathscr{W}}^{su}$, the set of $v \in T$ such that v_- belongs to the conical limit set $\Lambda_c\Gamma$ has full measure with respect to ν_T.

Remark (1). When Γ is torsion free, the image of the strong unstable foliation $\widetilde{\mathscr{W}}^{su}$ of $T^1\widetilde{M}$ by the covering map $T^1\widetilde{M} \to T^1M$ defines a foliation denoted by $\mathscr{W}^{su}$ on T^1M, the cocycle $c_{\widetilde{F}}$ defines by passing to the quotient a cocycle c_F for this foliation (already defined in Subsection 6.2, see Equation (80) and Equation (82)), hence we may work directly with c_F-quasi-invariant transverse measures for the foliation $\mathscr{W}^{su}$ on T^1M (defined as above by forgetting the equivariance property (ii)). The case with torsion (useful when working with arithmetic groups, for instance) makes it easier to work equivariantly on $T^1\widetilde{M}$ than to work with singular foliations on T^1M.

Remark (2). As seen in Subsection 3.9, for every $v \in T^1M$, the stable submanifold $T = W^s(v)$ of v in $T^1\widetilde{M}$ is a transversal to the strong unstable foliation $\widetilde{\mathscr{W}}^{su}$ of $T^1\widetilde{M}$, and every strong unstable leaf $W^{su}(w)$ with $w_- \neq v_+$ meets this transversal in one and only one point w'. If $U_T = \{w \in T^1\widetilde{M} \ : \ w_- \neq v_+\}$, then the map $\psi_T : U_T \to T$ defined by $w \mapsto w'$ is a continuous fibration, whose fiber above $w' \in T$ is the strong unstable leaf of w'.

In particular, any small enough transversal T' to the strong unstable foliation $\widetilde{\mathscr{W}}^{su}$ of $T^1\widetilde{M}$ admits a holonomy map $f : T' \to T$ where T is a submanifold of a stable submanifold. Hence by the above properties (i) and (iii), a Γ-equivariant $c_{\widetilde{F}}$-quasi-invariant transverse measure $\nu = (\nu_T)_{T\in\mathscr{T}(\widetilde{\mathscr{W}}^{su})}$ for $\widetilde{\mathscr{W}}^{su}$ is uniquely determined by the measures ν_T where T is a stable submanifold of $T^1\widetilde{M}$. We will use this remark without further comment in what follows.

Remark (3). Let us give another justification of our terminology of Gibbs measure, by further developping the analogy with symbolic dynamics. We refer to Subsection 3.8 for the notation of a countable alphabet A, the distance d on $\Sigma = A^{\mathbb{Z}}$, the full shift $\sigma : \Sigma \to \Sigma$, the cylinders $[a_{-n'}, \dots, a_n]$ where $n, n' \in \mathbb{N}$ and $a_{-n'}, \dots, a_n \in A$, a Hölder-continuous map $F : \Sigma \to \mathbb{R}$, and the definition of a Gibbs measure for the potential F on Σ (see Equation (40)).

For every $x = (x_i)_{i\in\mathbb{Z}} \in \Sigma$, define the *strong unstable leaf* of x as

$$W^{su}(x) = \{y = (y_i)_{i\in\mathbb{Z}} \in \Sigma \ : \ \exists\, n \in \mathbb{Z},\ \forall\, i \leq n, \quad y_i = x_i\} \ .$$

Note that these sets $W^{su}(x)$ for all $x \in \Sigma$ are either equal or disjoint. We define the *unstable Gibbs cocycle* of (Σ, σ, F) (compare with Equation (119), with [Rue3, Zin, Kel], and in particular with [HaR]) as the map c_F which associates, to all x, y in the same strong unstable leaf, the real number

$$c_F(x,y) = \sum_{i=1}^{+\infty} F(\sigma^{-i}x) - F(\sigma^{-i}y) ,$$

which exists by the Hölder-continuity of F, and satisfies the cocycle properties $c_F(x,y) + c_F(y,z) = c_F(x,z)$ and $c_F(x,y) = -c_F(y,x)$ if x, y, z are in the same strong unstable leaf. Similarly, let us define the *stable leaf* of x as

$$W^s(x) = \{y = (y_i)_{i\in\mathbb{Z}} \in \Sigma \;:\; \exists\, n, k \in \mathbb{Z},\ \forall\, i \geq n, \quad y_i = x_{i+k}\} .$$

We denote respectively by $\mathscr{W}^{su}$ and $\mathscr{W}^{s}$ the partitions of Σ in strong unstable leaves and in stable leaves.

The projection map $p : \Sigma \to \Sigma^+ = A^{\mathbb{N}}$ defined by $(x_i)_{i\in\mathbb{Z}} \mapsto (x_i)_{i\in\mathbb{N}}$ is a 1-Lipschitz fibration, whose fibers are contained in (strong) stable leaves. For all $z = (z_i)_{i\in\mathbb{N}}$ and $z' = (z'_i)_{i\in\mathbb{N}}$ in Σ^+, the map $f_{z',z} : p^{-1}(z) \to p^{-1}(z')$, uniquely defined by $(x_i)_{i\in\mathbb{Z}} \mapsto (y_i)_{i\in\mathbb{Z}}$ where $y_i = x_i$ for every $i < 0$, is a Lipschitz homeomorphism. Note that x and $f_{z',z}(x)$ are in the same strong unstable leaf, for every $x \in p^{-1}(z)$. Hence we will consider the family $(p^{-1}(z))_{z\in\Sigma^+}$ of fibers of p as a family of transversals to the partition into strong unstable leaves of Σ, and $(f_{z',z}))_{z,z'\in\Sigma^+}$ as a family of holonomy maps along the strong unstable leaves of Σ.

We define a *c_F-quasi-invariant transverse measure* for $\mathscr{W}^{su}$ to be a family $(\nu_z)_{z\in\Sigma^+}$, where ν_z is a measure on $p^{-1}(z)$ for every $z \in \Sigma^+$, such that there exists a constant $c > 0$ such that for all z, z' in Σ^+, with $f = f_{z',z}$, the measures $\nu_{z'}$ and $f_*\nu_z$ have the same measure class, and their Radon-Nikodym derivative satisfies, for ν_z-almost every $x \in p^{-1}(z)$,

$$\frac{1}{c}\, e^{c_F(f(x),\,x)} \leq \frac{d\,\nu_{z'}}{d\,f_*\nu_z}(f(x)) \leq c\; e^{c_F(f(x),\,x)} .$$

We will also consider ν_z as a measure on Σ with support in $p^{-1}(z)$.

By the cocycle property of the Radon-Nikodym derivatives, note that we may take $c = 1$, up to replacing the cocycle c_F by a cohomologous one.

Proposition 10.1 *Let m_F be a probability Gibbs measure for the potential F on Σ. Then there exists a c_F-quasi-invariant transverse measure $(\nu_z)_{z\in\Sigma^+}$ for $\mathscr{W}^{su}$ such that m_F disintegrates, by the fibration p, with family of conditional measures $(\nu_z)_{z\in\Sigma^+}$.*

When the alphabet A is finite, there exists in fact a unique, up a multiplicative constant, c_F-quasi-invariant transverse measure for $\mathscr{W}^{su}$, see [BoM, Prop. 3.3] when $F = 0$ (compare with [PoS, Prop. 6 (i)]) and [BaL, Prop. 4.4] for general F. Analogous uniqueness results for geodesic flows are the main goal of this chapter (see Remark (1) after Theorem 10.4).

Proof. We refer for instance to [Le] for basics on conditional measures and martingales.

Let $(\nu_z)_{z\in\Sigma^+}$ be a family of conditional measures of m_F for the fibration p. By definition, ν_z is a probability measure on $p^{-1}(z)$ such that for all Borel subsets B in Σ, the map $z \mapsto \nu_z(B)$ is measurable and we have

$$\int_{x\in\Sigma} \mathbb{1}_B(x)\, dm_F(x) = \int_{z\in\Sigma^+} \nu_z(B)\, d(p_*m_F)(z) . \tag{120}$$

Note that $(\nu_z)_{z\in\Sigma^+}$ is well defined up to a set of p_*m_F-measure zero of $z \in \Sigma^+$.

Lemma 10.2 *For p_*m_F-almost every $z = (z_i)_{i\in\mathbb{N}} \in \Sigma^+$, for all $a_{-n'}, \dots, a_{-1} \in A$, we have*

$$\nu_z([a_{-n'}, \dots, a_{-1}]) = \lim_{k\to+\infty} \frac{m_F([a_{-n'}, \dots, a_{-1}] \cap [z_0, \dots, z_k])}{m_F([z_0, \dots, z_k])} .$$

Proof. For every $k \in \mathbb{N}$, let $\mathscr{F}_k$ be the σ-algebra generated by the cylinders $[a_0, \dots, a_k]$ in Σ, where $a_0, \dots, a_k \in A$. Then $(\mathscr{F}_k)_{k\in\mathbb{N}}$ is a filtration in the Borel σ-algebra $\mathscr{B}$ of Σ, and a map $f : \Sigma \to \mathbb{R}$ is $\mathscr{F}_\infty$-measurable for $\mathscr{F}_\infty = \bigvee_{k\in\mathbb{N}} \mathscr{F}_k$ if and only if $f(x)$ depends only on $p(x)$ for all $x \in \Sigma$ and $z \mapsto f(p^{-1}(z)) : \Sigma^+ \to \mathbb{R}$ is measurable. For every Borel subset B in the probability space (Σ, m_F), the sequence $(X_k = E[\mathbb{1}_B \mid \mathscr{F}_k])_{k\in\mathbb{N}}$ is a (basic example of a) martingale adapted to the filtration $(\mathscr{F}_k)_{k\in\mathbb{N}}$: we have $E[X_{k+1} \mid \mathscr{F}_k] = X_k$ for every $k \in \mathbb{N}$. Since $\mathbb{1}_B$ is m_F-integrable, by the $\mathbb{L}^1$ martingale convergence theorem, the random variable X_n converges almost surely (and in $\mathbb{L}^1$) to $X_\infty = E[\mathbb{1}_B \mid \mathscr{F}_\infty]$. By the definition of the conditional expectation and by Equation (120), for m_F-almost every $x \in \Sigma$, we have $E[\mathbb{1}_B \mid \mathscr{F}_\infty](x) = \nu_{p(x)}(B)$. Since a Gibbs measure gives a nonzero mass to any cylinder (see Equation (40)) and since a map $f : \Sigma \to \mathbb{R}$ is $\mathscr{F}_k$-measurable if and only if $f(x)$ depends only on the components $x_0, \dots, x_k$ of x, for m_F-almost every $x = (x_i)_{i\in\mathbb{Z}} \in \Sigma$, we have $E[\mathbb{1}_B \mid \mathscr{F}_k](x) = \frac{m_F(B\cap[x_0, \dots, x_k])}{m_F([x_0, \dots, x_k])}$. The result follows. □

For all sets X and maps $g, g' : X \to \mathbb{R}$, let us write $g \asymp g'$ if there exists $c > 0$ such that $\frac{1}{c}\, g(x) \le g'(x) \le c\, g(x)$ for every $x \in X$. The above lemma and Equation (40) imply that there exists $\delta'_F \in \mathbb{R}$ such that for m_F-almost every $x = (x_i)_{i\in\mathbb{Z}} \in \Sigma$, we have

$$\nu_{p(x)}([x_{-n'}, \dots, x_{-1}]) \asymp \lim_{n\to+\infty} \frac{e^{\sum_{i=-n'}^{n} F(\sigma^i x) - (n+n'+1)\delta'_F}}{e^{\sum_{i=0}^{n} F(\sigma^i x) - (n+1)\delta'_F}} = e^{\sum_{i=1}^{n'} F(\sigma^{-i}x) - n'\,\delta'_F} .$$

Hence for p_*m_F-almost all $z, z' \in \Sigma^+$ and for ν_z-almost every $x = (x_i)_{i\in\mathbb{Z}} \in p^{-1}(z)$, if $f = f_{z',\,z}$, then

$$\frac{\nu_{z'}([x_{-n'}, \dots, x_{-1}])}{\nu_z([x_{-n'}, \dots, x_{-1}])} \asymp e^{\sum_{i=1}^{n'} F(\sigma^{-i} f(x)) - F(\sigma^{-i}x)} .$$

Proposition 10.1 now follows by taking limits and modifying the family $(\nu_z)_{z\in\Sigma^+}$ on a set of p_*m_F-measure zero. □

10.2 Quasi-invariant measures on the space of horospheres

Let us start this subsection by giving a few definitions. Given a topological space X endowed with a continuous action of a discrete group G, a *real cocycle* on X is a continuous map $\check{c} : G \times X \to \mathbb{R}$ such that, for all $x \in X$ and $\gamma, \gamma' \in G$, we have

$$\check{c}\,(\gamma\gamma', x) = \check{c}\,(\gamma, \gamma' x) + \check{c}\,(\gamma', x) .$$

A real cocycle $\check{c}\,'$ on X is *cohomologous* to $\check{c}$ via a continuous map $f : X \to \mathbb{R}$ if $\check{c}\,'(\gamma, x) - \check{c}(\gamma, x) = f(\gamma x) - f(x)$ for all $x \in X$ and $\gamma \in G$. Given a real cocycle $\check{c}$ on X, a *$\check{c}$-quasi-invariant measure* on X is a (not necessarily finite) locally finite nonzero (positive Borel) measure $\check{\nu}$ on X such that for every $\gamma \in G$, the measures $\gamma_*\check{\nu}$ and $\check{\nu}$ are equivalent, with, for every $x \in X$,

$$\frac{d\gamma_*\check{\nu}}{d\check{\nu}}(x) = e^{\check{c}\,(\gamma^{-1}, x)} .$$

We will be interested in the case when X is the space $\mathscr{H}_{\widetilde{M}}$ of horospheres of $\widetilde{M}$, endowed with the topology of Hausdorff convergence on compact subsets and the action of $G = \Gamma$ on subsets of $\widetilde{M}$, and when $\breve{c}$ is the real cocycle

$$\breve{c}_{\widetilde{F}} = \breve{c}_{\widetilde{F},\,x_0} : (\gamma, H) \mapsto C_{F\circ\iota,\,H_\infty}(x_0, \gamma^{-1}x_0)\ ,$$

where H is a horosphere centered at H_∞ and $\gamma \in \Gamma$.

Note that $\breve{c}_{\widetilde{F}} = 0$ if $\widetilde{F} = 0$, that $\breve{c}_{\widetilde{F}} = \breve{c}_{\widetilde{F},\,x_0}$ does depend on x_0, but that $\breve{c}_{\widetilde{F},\,x_0'}$ is cohomologous to $\breve{c}_{\widetilde{F},\,x_0}$ via the map $f : H \mapsto C_{F\circ\iota,\,H_\infty}(x_0', x_0)$ for every $x_0' \in \widetilde{M}$. If two potentials $\widetilde{F}^*$ and $\widetilde{F}$ are cohomologous via the map $\widetilde{G}$, then $\widetilde{F}^*\circ\iota$ and $\widetilde{F}\circ\iota$ are cohomologous via the map $-\widetilde{G}\circ i$, hence, by Equation (23), the real cocycle $\breve{c}_{\widetilde{F}^*}$ is cohomologous to $\breve{c}_{\widetilde{F}}$ via the map $f : H \mapsto \widetilde{G}\circ\iota(v_{x_0\,H_\infty})$, with the notation of Remark (1) just before Lemma 3.4.

As explained for instance in [Sch1, Sch3], there exists a correspondance between Γ-equivariant $c_{\widetilde{F}}$-quasi-invariant transverse measures $\nu = (\nu_T)_{T\in\mathscr{T}(\widetilde{\mathscr{W}}^{su})}$ for $\widetilde{\mathscr{W}}^{su}$ and $\breve{c}_{\widetilde{F}}$-quasi-invariant measures $\breve{\nu}$ on $\mathscr{H}_{\widetilde{M}}$, see Proposition 10.3 below. This second viewpoint is the one used in [Rob1] when $F = 0$, in which case the $\breve{c}_{\widetilde{F},\,x_0}$-quasi-invariant measures on $\mathscr{H}_{\widetilde{M}}$ are just the Γ-equivariant measures, which explains why it is easier to work with the second viewpoint then. The first viewpoint turns out to be a bit more convenient when F is not constant.

Let $\mathscr{H}^{su}$ be the space of strong unstable leaves of $T^1\widetilde{M}$, that is, the quotient topological space of $T^1\widetilde{M}$ by the equivalence relation "being in the same strong unstable leaf", with the quotient action of Γ. We denote by $p^{su} : T^1\widetilde{M} \to \mathscr{H}^{su}$ the canonical projection. Note that the map associating to a horosphere in $\widetilde{M}$ its outer unit normal bundle is a Γ-equivariant homeomorphism between the space of horospheres $\mathscr{H}_{\widetilde{M}}$ in $\widetilde{M}$ and $\mathscr{H}^{su}$, whose inverse is the map (also denoted by π) which associates to a strong unstable leaf W the horosphere $\pi(W)$. For every $v \in T^1\widetilde{M}$, the restriction of $\pi\circ p^{su}$ to the transversal $T = W^s(v)$ to the foliation $\widetilde{\mathscr{W}}^{su}$ in $T^1\widetilde{M}$ is a homeomorphism onto its image in $\mathscr{H}_{\widetilde{M}}$, and we will identify T with its image by this map.

Let us endow $\partial_\infty\widetilde{M}\times\mathbb{R}$ with the action of Γ defined as follows: for all $\gamma\in\Gamma$, $\xi\in\partial_\infty\widetilde{M}$, and $t\in\mathbb{R}$,

$$\gamma(\xi, t) = (\gamma\xi, t + \beta_\xi(\gamma^{-1}x_0, x_0))\ .$$

Note that for all horospheres H with center H_∞ and $u \in H$, the real number $\beta_{H_\infty}(x_0, u)$ does not depend on $u \in H$, it will be denoted by $\beta_{H_\infty}(x_0, H)$. The map from $\mathscr{H}_{\widetilde{M}}$ to $\partial_\infty\widetilde{M}\times\mathbb{R}$, defined by

$$H \mapsto \big(H_\infty, \beta_{H_\infty}(x_0, H)\big)\ ,$$

is a Γ-equivariant homeomorphism. It depends on x_0, but only up to a translation (depending on the first factor) in the second factor $\mathbb{R}$.

Proposition 10.3 *For every Γ-equivariant $c_{\widetilde{F}}$-quasi-invariant transverse measure $\nu = (\nu_T)_{T\in\mathscr{T}(\widetilde{\mathscr{W}}^{su})}$ for $\widetilde{\mathscr{W}}^{su}$, there exists one and only one $\breve{c}_{\widetilde{F}}$-quasi-invariant measure $\breve{\nu} = \breve{\nu}_{x_0}$ on $\mathscr{H}_{\widetilde{M}}$ such that for every $v \in T^1\widetilde{M}$, if $T = W^s(v)$, then for all $w \in T$ and $H = \pi(W^{su}(w))$, we have*

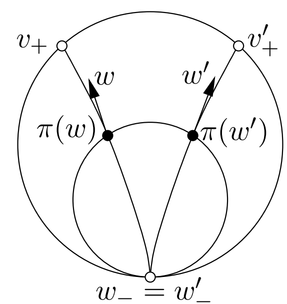
$$d\breve{\nu}(H) = e^{C_{F\circ\iota,\,w_-}(\pi(w),\,x_0)}\, d\nu_T(w)\ . \tag{121}$$

Conversely, for every $\check{c}_{\widetilde{F}}$-quasi-invariant measure $\check{\nu}$ on $\mathscr{H}_{\widetilde{M}}$, there exists one and only one Γ-equivariant $c_{\widetilde{F}}$-quasi-invariant transverse measure $\nu = (\nu_T)_{T\in\mathscr{T}(\widetilde{\mathscr{W}}^{su})}$ for $\widetilde{\mathscr{W}}^{su}$ such that Equation (121) holds for every stable leaf T.

Note that the measure $\check{\nu} = \check{\nu}_{x_0}$ does depend on the base point x_0, hence the aforementioned correspondance is not canonical.

Note that the above measure $\check{\nu}$ on $\mathscr{H}_{\widetilde{M}}$ gives full measure to the set of horospheres centered at conical limit points if and only if the transverse measure ν for $\widetilde{\mathscr{W}}^{su}$ gives full measure to the negatively recurrent set.

Proof. The open subsets $T = W^s(v)$ of $\mathscr{H}_{\widetilde{M}}$ cover $\mathscr{H}_{\widetilde{M}}$ as v ranges over $T^1\widetilde{M}$. Let us prove that the measures on these open subsets defined by the member on the right of Equation (121) glue together to define a measure on $\mathscr{H}_{\widetilde{M}}$, that is, that two of them coincide on the intersection of their domain.



Let $v, v' \in T^1\widetilde{M}$, and let $T = W^s(v)$ and $T' = W^s(v')$. The map from $\{w \in T \;:\; w_- \neq v'_+\}$ to $\{w' \in T' \;:\; w'_- \neq v_+\}$ which sends w to the unique element $w' \in W^{su}(w)$ such that $w'_+ = v'_+$ is a holonomy map for the foliation $\widetilde{\mathscr{W}}^{su}$. Note that $w'_- = w_-$ and that $\pi(w)$ and $\pi(w')$ lie on the same horosphere centered at $w'_- = w_-$. In particular, $\beta_{w_-}(\pi(w), \pi(w') = 0$.

By the properties (i) and (iii) of ν, we hence have, for every $w \in \{w \in T \;:\; w_- \neq v'_+\}$,

$$\begin{aligned} e^{C_{F\circ\iota,\, w'_-}(\pi(w'),\, x_0)}\; d\nu_{T'}(w') &= e^{C_{F\circ\iota,\, w'_-}(\pi(w'),\, x_0)}\; e^{c_{\widetilde{F}}(w',\, w)}\; d\nu_T(w) \\ &= e^{C_{F\circ\iota,\, w_-}(\pi(w'),\, x_0) - C_{F\circ\iota,\, w_-}(\pi(w'),\, \pi(w))}\; d\nu_T(w) \\ &= e^{C_{F\circ\iota,\, w_-}(\pi(w),\, x_0)}\; d\nu_T(w)\;. \end{aligned} \tag{122}$$

This proves the existence and uniqueness of a nonzero measure $\check{\nu}$ on $\mathscr{H}_{\widetilde{M}}$ satisfying Equation (121) for every stable leaf T.

Let us now prove that $\check{\nu}$ is $\check{c}_{\widetilde{F}}$-quasi-invariant. Let $\gamma \in \Gamma$, $v \in T^1\widetilde{M}$, and $T = W^s(v)$, so that $\gamma T = W^s(\gamma v)$. Let $w \in \{u \in \gamma T \;:\; u_- \neq v_+\}$ and $H = \pi(W^{ss}(w))$. Using respectively

- the definition of $\check{\nu}$ for the second equality,
- the invariance of the Gibbs cocycle (see Equation (22)) and the property (ii) of ν for the third equality,
- the property (iii) of ν for the holonomy map from $\{u \in T \;:\; u_- \neq \gamma v_+\}$ to $\{u \in \gamma T \;:\; u_- \neq v_+\}$ for the foliation $\widetilde{\mathscr{W}}^{su}$ sending w' to the unique element $w \in W^{su}(w')$ such that $w_+ = \gamma v_+$ for the fourth equality,
- the fact that $w'_- = w_-$ and the definition of the cocycle $c_{\widetilde{F}}$ for the fifth equality,
- the cocycle properties (see Equation (21)) of the Gibbs cocycle for the sixth equality,
- the definition of the cocycle $\check{c}_{\widetilde{F}}$ and the facts that $\pi(W^{su}(w')) = \pi(W^{su}(w)) = H$

and $H_\infty = w'_-$ for the seventh equality, we have

$$\begin{aligned}
d\gamma_*\check\nu(H) &= d\check\nu(\gamma^{-1}H) = e^{C_{F\circ\iota,\,(\gamma^{-1}w)_-}(\pi(\gamma^{-1}w),\,x_0)}\;d\nu_T(\gamma^{-1}w)\\
&= e^{C_{F\circ\iota,\,w_-}(\pi(w),\,\gamma x_0)}\;d\nu_{\gamma T}(w) = e^{C_{F\circ\iota,\,w_-}(\pi(w),\,\gamma x_0)}\;e^{c_{\widetilde F}(w,\,w')}\;d\nu_T(w')\\
&= e^{C_{F\circ\iota,\,w'_-}(\pi(w),\,\gamma x_0)-C_{F\circ\iota,\,w'_-}(\pi(w),\,\pi(w'))}\;d\nu_T(w')\\
&= e^{C_{F\circ\iota,\,w'_-}(x_0,\,\gamma x_0)}\;e^{C_{F\circ\iota,\,w'_-}(\pi(w'),\,x_0)}\;d\nu_T(w')\\
&= e^{\check c_{\widetilde F}(\gamma^{-1},\,H)}\;d\check\nu(H)\;.
\end{aligned}\tag{123}$$

This proves that $\check\nu$ is $\check c_{\widetilde F}$-quasi-invariant.

Conversely, let $\check\nu$ be a $\check c_{\widetilde F}$-quasi-invariant nonzero measure on $\mathscr{H}_{\widetilde M}$. For every unstable leaf $T = W^s(v)$, let us define $d\nu_T(w) = e^{-C_{F\circ\iota,\,w_-}(\pi(w),\,x_0)}\;d\check\nu(w)$. The computations (122) and (123) show that it is possible to define, using restrictions and holonomy maps, a measure ν_T for any transversal T to $\widetilde{\mathscr{W}}^{su}$ such that the family $\nu = (\nu_T)_{T\in\mathscr{T}(\widetilde{\mathscr{W}}^{su})}$ satisfies the properties (i), (ii) and (iii). □

10.3 Classification of quasi-invariant transverse measures for $\mathscr{W}^{su}$

Let us now state our unique ergodicity result of the strong unstable foliation, starting by defining the transverse measure which will play the central role.

By Subsection 3.9, if $\delta = \delta_{\Gamma,\,F} < +\infty$, to every Patterson density $(\mu^\iota_x)_{x\in\widetilde M}$ of dimension δ for $(\Gamma, F\circ\iota)$ is associated, for any $v \in T^1\widetilde M$ with stable leaf $T = W^s(v)$, a nonzero measure μ^ι_T on T, defined (independently of x_0), using the homeomorphism from T to $(\partial_\infty\widetilde M - \{v_+\})\times\mathbb{R}$ defined by $w' \mapsto \big(w'_-, t = \beta_{v_+}(\pi(v), \pi(w'))\big)$, by (see Equation (53))

$$d\mu^\iota_T(w') = e^{C_{F\circ\iota-\delta,\,w'_-}(x_0,\,\pi(w'))}\;d\mu^\iota_{x_0}(w'_-)\,dt\;.\tag{124}$$

This measure is finite on the compact subsets of T, and if $\mu^\iota_{x_0}({}^c\Lambda_c\Gamma) = 0$, then the set of $w' \in T$ such that w'_- belongs to $\Lambda_c\Gamma$ has full measure with respect to μ^ι_T. By Remark (2) in Subsection 10.1, by Equation (54) and Equation (55), there exists hence a unique Γ-equivariant $c_{\widetilde F}$-quasi-invariant transverse measure $(\mu^\iota_T)_{T\in\mathscr{T}(\widetilde{\mathscr{W}}^{su})}$ for $\widetilde{\mathscr{W}}^{su}$, giving full measure to the positively recurrent set, such that if $T = W^s(v)$ for some $v \in T^1\widetilde M$, then μ^ι_T is the above measure.

The next result proves that such a transverse measure is unique, under our usual assumptions (see Chapter 8, in particular Theorem 8.1, for situations where they are satisfied).

Theorem 10.4 *Let $\widetilde M$ be a complete simply connected Riemannian manifold, with dimension at least 2 and pinched negative curvature at most -1. Let Γ be a nonelementary discrete group of isometries of $\widetilde M$. Let $\widetilde F : \widetilde M \to \mathbb{R}$ be a Γ-invariant Hölder-continuous map. Assume that $\delta_{\Gamma,\,F}$ is finite and positive, and that the Gibbs measure m_F on T^1M is finite and mixing under the geodesic flow.*

Then the family $(\mu^\iota_T)_{T\in\mathscr{T}(\widetilde{\mathscr{W}}^{su})}$ is, up to a multiplicative constant, the unique Γ-equivariant $c_{\widetilde F}$-quasi-invariant transverse measure, giving full measure to the negatively recurrent set, for the strong unstable foliation on $T^1\widetilde M$. Furthermore, $(\mu^\iota_T)_{T\in\mathscr{T}(\widetilde{\mathscr{W}}^{su})}$ is ergodic.

Remark (1). If $F = 0$, this theorem (with the formulation of Corollary 10.5 in the next remark if M has dimension 2) is due to Furstenberg [Fur] when M is a compact hyperbolic surface, to Dani [Dan], Dani-Smillie [DaS], Ratner [Rat3] when M is a finite volume hyperbolic surface, to Bowen-Marcus [BoM] when M is a compact or convex-cocompact manifold, via a coding argument (see also Coudene [Cou1] for a very short dynamical argument in the case of finite volume surfaces), and to Roblin [Rob1, Chap. 6] in full generality. When $F = 0$, $\widetilde{M}$ is a higher rank symmetric space of noncompact type, G is the identity component of the isometry group of $\widetilde{M}$, and Γ is a lattice in G, we refer to Ratner's classification theorem of the measures on $\Gamma\backslash G$ invariant under a unipotent subgroup of G (see [Rat1, Rat2, MaT, Mor]). It would be interesting to study the case F nonconstant in higher rank.

For general potentials F (with the slightly different definition of Gibbs measure mentioned in the beginning of Chapter 3), this theorem was proven in Babillot-Ledrappier [BaL] in a symbolic framework or when M is an abelian Riemannian cover of a compact manifold, and in Schapira [Sch3] when M is compact or convex-cocompact. This theorem (with Γ torsion free and $\widetilde{F}$ bounded) is stated, and its proof is sketched, in [Sch1, Chap. 8.2.2] following Roblin's arguments in [Rob1, Chap. 6]. In the proof below, we will also follow closely Roblin's strategy of proof, though the presence of potentials requires many adjustements and new lemmas.

Remark (2). When $\widetilde{M}$ is a surface, it is well-known since Furstenberg's work (see for instance [Sch2]) that this theorem corresponds to a unique ergodicity result for horocyclic flows. When $\widetilde{M}$ is a symmetric space, this theorem corresponds to a unique ergodicity result for some unipotent group actions.

More precisely, assume first that $\widetilde{M}$ is a surface, fix one of the two orientations of $\widetilde{M}$ and assume that Γ preserves the orientation. Then each horosphere is naturally oriented, say anticlockwise. A *horocyclic flow* on $T^1\widetilde{M}$ is a continuous one-parameter group of homeomorphisms $(h_s)_{s\in\mathbb{R}}$ of $T^1\widetilde{M}$, such that, for every $v \in T^1\widetilde{M}$, the map $s \mapsto h_s v$ is an orientation preserving homeomorphism from $\mathbb{R}$ to $W^{su}(v)$ and for every $s \in \mathbb{R}$, the homeomorphism $v \mapsto h_s v$ of $T^1\widetilde{M}$ is Γ-equivariant. By passing to the quotient, it induces a continuous one-parameter group of homeomorphisms of T^1M, called the *horocyclic flow* on T^1M and also denoted by $(h_s)_{s\in\mathbb{R}}$. One way to define $h_s v$ is as the element of $W^{su}(v)$ on the right of v if $s \geq 0$, and on its left if $s \leq 0$, such that the Riemannian length of the arc of the horocycle $\pi(W^{su}(v))$ from $\pi(v)$ to $\pi(h_s(v))$ is s. In constant curvature -1, this horocycle flow satisfies furthermore that $\phi_t \circ h_s = h_{s\,e^t} \circ \phi_s$ for all $s, t \in \mathbb{R}$. But in variable curvature, this is no longer always true, and the Riemannian parametrisation is not always the appropriate one. See for instance [Marc] which constructs, when M is compact, a horocycle flow satisfying, besides more precise regularity properties, that $\phi_t \circ h_s = h_{s\,e^{\delta_\Gamma t}} \circ \phi_s$ for all $s, t \in \mathbb{R}$.

A locally finite (positive Borel) measure m on $T^1\widetilde{M}$ is $c_{\widetilde{F}}$*-quasi-invariant under the horocyclic flow* $(h_s)_{s\in\mathbb{R}}$ if for every $s \in \mathbb{R}$, the measures m and $(h_s)_*m$ are mutually absolutely continuous, and satisfy, for every $v \in T^1\widetilde{M}$,

$$\frac{d\,(h_s)_*m}{d\,m}(h_s v) = e^{c_{\widetilde{F}}(v,\, h_s v)}\ .$$

More generally, assume that there exists a continuous action (on the left) of a unimodular real Lie group $\mathbb{U}$ on $T^1\widetilde{M}$, commuting with the action of Γ, such that for every

$v \in T^1\widetilde{M}$ the map $u \mapsto u \cdot v$ is a homeomorphism from $\mathbb{U}$ to $W^{su}(v)$. Such an action exists for instance when $\widetilde{M}$ is a symmetric space, that is, a real, complex, quaternionic or octonionic hyperbolic space (see for instance [Park]), since a nilpotent Lie group is unimodular. We say that a locally finite measure m on $T^1\widetilde{M}$ is *$c_{\widetilde{F}}$-quasi-invariant under the action of* $\mathbb{U}$ if for every $u \in \mathbb{U}$, the measures m and $u_* m$ are absolutely continuous one with respect to each other, and satisfy, for every $v \in T^1\widetilde{M}$,

$$\frac{d\,u_* m}{d\,m}(u \cdot v) = e^{c_{\widetilde{F}}(v,\, u\cdot v)}\ .$$

Corollary 10.5 *Under the assumptions of Theorem 10.4, there exists, up to a multiplicative constant, a unique locally finite Γ-invariant measure on $T^1\widetilde{M}$, giving full measure to $\{w \in T^1\widetilde{M}\ :\ w_- \in \Lambda_c\Gamma\}$, which is $c_{\widetilde{F}}$-quasi-invariant under the action of $\mathbb{U}$.*

The special case when $\mathbb{U} = \mathbb{R}$ gives that if $\widetilde{M}$ has dimension 2 and if $(h_s)_{s\in\mathbb{R}}$ is a horocyclic flow, then under the assumptions of Theorem 10.4, there exists, up to a multiplicative constant, a unique locally finite Γ-invariant measure on $T^1\widetilde{M}$, giving full measure to $\{w \in T^1\widetilde{M}\ :\ w_- \in \Lambda_c\Gamma\}$, which is $c_{\widetilde{F}}$-quasi-invariant under $(h_s)_{s\in\mathbb{R}}$.

Proof. We start the proof by a construction which, when $\mathbb{U} = \mathbb{R}$, is due to Burger [Bur] (see more generally [Rob1, §1C]) when $F = 0$, and to the last author [Sch2, Sch3] (see also Lemma 10.7 below) when Γ is torsion free, with a local approach.

Recall (see Subsection 3.9 before Proposition 3.17, exchanging the stable and unstable foliations) that, for every $v \in T^1\widetilde{M}$, with $T = W^s(v)$, the map ψ_T from the open set $U_T = \{w \in T^1\widetilde{M}\ \ :\ \ w_- \neq v_+\}$ to T, sending $w \in U_T$ to the unique element w' in $W^{su}(w) \cap T$, is a continuous fibration over T, whose fiber over $w' \in T$ is the strong unstable leaf of w'. (Note that U_T is invariant under $\mathbb{U}$, and this fibration is a principal fibration under $\mathbb{U}$.)

Let λ be a Haar measure on $\mathbb{U}$ (say the usual Lebesgue measure when $\mathbb{U} = \mathbb{R}$) and, for every $v \in T^1\widetilde{M}$, let $\lambda_{W^{su}(v)}$ be the pushforward measure of λ by the map $u \mapsto u \cdot v$, whose support is $W^{su}(v)$. Since $\mathbb{U}$ acts transitively on $W^{su}(v)$ and since λ is invariant under right translations, the measure $\lambda_{W^{su}(v)}$ indeed depends only on the strong unstable leaf of v. Since λ is invariant under left translations, we have $u_*\lambda_{W^{su}(v)} = \lambda_{W^{su}(v)}$ for every $u \in \mathbb{U}$. Since the action of Γ on $T^1\widetilde{M}$ commutes with the action of $\mathbb{U}$, for every $\gamma \in \Gamma$, we have $\gamma_*\lambda_{W^{su}(v)} = \lambda_{W^{su}(\gamma v)}$. Note that since the action is continuous, for every $v \in T^1\widetilde{M}$, with $T = W^s(v)$, for every $f \in \mathscr{C}_c(T^1\widetilde{M};\mathbb{R})$ with support contained in U_T, the map from T to $\mathbb{R}$ defined by

$$w' \mapsto \int_{w\in W^{su}(w')} f(w)\, e^{c_{\widetilde{F}}(w,\, w')}\, d\lambda_{W^{su}(w')}(w) = \int_{u\in\mathbb{U}} f(u\cdot w')\, e^{c_{\widetilde{F}}(u\cdot w',\, w')}\, d\lambda(u)$$

is continuous with compact support.

Let $\nu = (\nu_T)_{T\in\mathscr{T}(\widetilde{\mathscr{W}}^{su})}$ be a $c_{\widetilde{F}}$-quasi-invariant transverse measure for $\widetilde{\mathscr{W}}^{su}$. We claim that there exists a unique (positive Borel) measure $\widetilde{m}^{\nu,\lambda}$ on $T^1\widetilde{M}$ such that for every $v \in T^1\widetilde{M}$, with $T = W^s(v)$, the restriction to U_T of the measure $\widetilde{m}^{\nu,\lambda}$ disintegrates by the fibration ψ_T over the measure ν_T, with conditional measure on the fiber $W^{su}(w')$ of $w' \in T$ the measure $e^{c_{\widetilde{F}}(w,\, w')}\, d\lambda_{W^{su}(w')}(w)$: for every $f \in \mathscr{C}_c(T^1\widetilde{M};\mathbb{R})$ with support contained in

U_T, we have

$$\int_{w\in T^1\widetilde{M}} f(w)\, d\widetilde{m}^{\nu,\lambda}(w) = \int_{w'\in T}\int_{u\in\mathbb{U}} f(u\cdot w')\, e^{c_{\widetilde{F}}(u\cdot w',\,w')}\, d\lambda(u)\, d\nu_T(w')\;.$$

We refer to the proof of Lemma 10.7 for a proof of this claim.

Lemma 10.6 *The map $\nu \mapsto \widetilde{m}^{\nu,\lambda}$ is a bijection from the set of Γ-equivariant $c_{\widetilde{F}}$-quasi-invariant transverse measures for $\widetilde{\mathscr{W}}^{su}$, which give full measure to the negatively recurrent set, to the set of Γ-invariant locally finite measures on $T^1\widetilde{M}$ which are $c_{\widetilde{F}}$-quasi-invariant under the action of $\mathbb{U}$ and give full measure to $\{w\in T^1\widetilde{M}\;:\; w_-\in\Lambda_c\Gamma\}$.*

Proof. By construction, ν is Γ-equivariant if and only if $\widetilde{m}^{\nu,\lambda}$ is Γ-invariant, and ν_T is locally finite for every T if and only if $\widetilde{m}^{\nu,\lambda}$ is locally finite. For every $v\in T^1\widetilde{M}$, with $T=W^s(v)$, since the support of $\lambda_{W^{su}(v)}$ is equal to $W^{su}(v)$, the measure ν_T gives full measure to $\{w'\in T\;:\; w'_-\in\Lambda_c\Gamma\}$ if and only if the restriction of $\widetilde{m}^{\nu,\lambda}$ to U_T gives full measure to $\{w\in U_T\;:\; w_-\in\Lambda_c\Gamma\}$.

For all $v\in T^1\widetilde{M}$, $w\in T=W^s(v)$ and $u\in\mathbb{U}$, since $\lambda_{W^{su}(w')}$ is invariant under $\mathbb{U}$ and by the cocycle property of $c_{\widetilde{F}}$, we have

$$d\widetilde{m}^{\nu,\lambda}(u\cdot w) = \int_{w'\in T} e^{c_{\widetilde{F}}(u\cdot w,\,w')}\, d\lambda_{W^{su}(w')}(u\cdot w)\, d\nu_T(w') = e^{c_{\widetilde{F}}(u\cdot w,\,w)}\, d\widetilde{m}^{\nu,\lambda}(w)\;.$$

Hence $m^{\nu,\lambda}$ is $c_{\widetilde{F}}$-quasi-invariant under the action of $\mathbb{U}$.

Conversely, let $\widetilde{m}$ be a Γ-invariant locally finite measure on $T^1\widetilde{M}$ which is $c_{\widetilde{F}}$-quasi-invariant under the action of $\mathbb{U}$. Then given any $v\in T^1\widetilde{M}$, with $T=W^s(v)$, for the principal fibration $U_T\to T$ with group $\mathbb{U}$, the conditional measures of $m_{|U_T}$ on the fiber $W^{su}(w')=\mathbb{U}\cdot w'$ of almost every $w'\in T$ are invariant under $\mathbb{U}$. This conditional measure may be taken to be equal to $\lambda_{W^{su}(w')}$, by the uniqueness property of Haar measures, up to multiplying by a measurable function the (locally finite) measure ν_T on T over which $\widetilde{m}$ disintegrates. Then the family $\nu=(\nu_T)_{T\in\mathscr{T}(\widetilde{\mathscr{W}}^{su})}$ is clearly a $c_{\widetilde{F}}$-quasi-invariant transverse measure for $\widetilde{\mathscr{W}}^{su}$. $\square$

Since for every $t>0$, we have $\widetilde{m}^{t\nu,\lambda}=t\,\widetilde{m}^{\nu,\lambda}$, Corollary 10.5 indeed follows from Theorem 10.4. $\square$

Proof of Theorem 10.4. Let $(\mu^\iota_x)_{x\in\widetilde{M}}$ and $(\mu_x)_{x\in\widetilde{M}}$ be the Patterson densities of the same dimension δ for $(\Gamma,F\circ\iota)$ and (Γ,F), such that m_F is the Gibbs measure on T^1M (induced by the Gibbs measure $\widetilde{m}_F$ on $T^1\widetilde{M}$) associated with this pair of Patterson densities. Note that they are uniquely defined up to a scalar multiple, and give full measure to the conical limit set, by Corollary 5.14 and Corollary 5.12. Hence the family $\mu^\iota=(\mu^\iota_T)_{T\in\mathscr{T}(\widetilde{\mathscr{W}}^{su})}$ associated to $(\mu^\iota_x)_{x\in\widetilde{M}}$ by Equation (124) is indeed a Γ-equivariant $c_{\widetilde{F}}$-quasi-invariant transverse measure for $\widetilde{\mathscr{W}}^{su}$, giving full measure to the negatively recurrent set.

The scheme of the eight steps proof of Theorem 10.4 is the following one. Given a second such family ν, we first construct a measure $m^{\nu,\mu}$ which is a quasi-product measure of the measures defined by ν on the stable leaves and the measures defined by m_F on the strong unstable leaves, so that $m_F=m^{\mu^\iota,\mu}$. We then introduce diffusion operators $(I_r)_{r>0}$ along the strong unstable leaves acting on the locally integrable functions on T^1M.

We use the mixing property of the geodesic flow (see Subsection 8.1) combined with the action of these diffusion operators to prove that $m^{\nu,\,\mu}$ is absolutely continuous with respect to $m_F = m^{\mu^\iota,\,\nu}$ (this is the most technical part, which will require several steps). We then conclude that ν and μ^ι are homothetic by an easy argument of disintegration and ergodicity.

Step 1. Let $\nu = (\nu_T)_{T\in\mathscr{T}(\widetilde{\mathscr{W}}^{su})}$ be a Γ-equivariant $c_{\widetilde{F}}$-quasi-invariant transverse measure for $\widetilde{\mathscr{W}}^{su}$, giving full measure to the negatively recurrent set. In this first step, we construct a measure $\widetilde{m}^{\nu,\,\mu}$ on $T^1\widetilde{M}$ which, for every stable leaf T, disintegrates, with conditional measures in the class of the strong unstable measures $\mu_{W^{su}(w')}$ (as does the Gibbs measure), over the measure ν_T (instead of over the measure μ^ι_T as does the Gibbs measure).

Recall (see Subsection 3.9 before Proposition 3.17, exchanging the stable and unstable foliations) that, for every $v \in T^1\widetilde{M}$, with $T = W^s(v)$, the map ψ_T from the open set $U_T = \{w \in T^1\widetilde{M} \;:\; w_- \neq v_+\}$ to T, sending w to the unique element w' in $W^{su}(w)\cap T$, is a continuous fibration over T, whose fiber over $w' \in T$ is the strong unstable leaf of w'.

Lemma 10.7 *There exists a unique (positive Borel) measure $\widetilde{m}^{\nu,\,\mu}$ on $T^1\widetilde{M}$ such that for every $v \in T^1\widetilde{M}$, with $T = W^s(v)$, the restriction to U_T of the measure $\widetilde{m}^{\nu,\,\mu}$ disintegrates by the fibration ψ_T over the measure ν_T, with conditional measure $e^{c_{\widetilde{F}}(w,\,w')}\,d\mu_{W^{su}(w')}(w)$ on the fiber $W^{su}(w')$ of $w' \in T$: for every $w \in T^1\widetilde{M}$ such that $w_- \neq v_+$, we have*

$$d\widetilde{m}^{\nu,\,\mu}(w) = \int_{w'\in T} e^{c_{\widetilde{F}}(w,\,w')}\,d\mu_{W^{su}(w')}(w)\,d\nu_T(w')\,. \tag{125}$$

Furthermore, $\widetilde{m}^{\nu,\,\mu}$ is Γ-invariant and locally finite, and the set $\widetilde{\Omega}_c\Gamma$ of elements $v \in T^1\widetilde{M}$ such that $v_-, v_+ \in \Lambda_c\Gamma$ has full measure for $\widetilde{m}^{\nu,\,\mu}$.

Hence the measure $\widetilde{m}^{\nu,\,\mu}$ defines a locally finite measure $m^{\nu,\,\mu}$ on T^1M, which gives full measure to the set of two-sided recurrent elements.

Note that by the disintegration of the Gibbs measure (see Proposition 3.17 (1), exchanging the stable and unstable foliations), we have

$$\widetilde{m}_F = \widetilde{m}^{\mu^\iota,\,\mu} \quad\text{and}\quad m_F = m^{\mu^\iota,\,\mu}\,.$$

Proof. Let $v, v' \in T^1\widetilde{M}$, and $T = W^s(v)$, $T' = W^s(v')$. The map from $U_{T'}\cap T = \{w' \in T \;:\; w'_- \neq v'_+\}$ to $U_T \cap T' = \{w'' \in T' \;:\; w''_- \neq v_+\}$ sending w' to the unique element w'' of $W^{su}(w')\cap T'$ is a holonomy map for the strong unstable foliation. Hence $d\nu_{T'}(w'') = e^{c_{\widetilde{F}}(w'',\,w')}d\nu_T(w')$ by the $c_{\widetilde{F}}$-quasi-invariance property of ν. Since $W^{su}(w') = W^{su}(w'')$ and by the cocycle property of $c_{\widetilde{F}}$, for every $w \in U_T\cap U_{T'}$, we therefore have

$$\int_{w'\in T} e^{c_{\widetilde{F}}(w,\,w')}\,d\mu_{W^{su}(w')}(w)\,d\nu_T(w') = \int_{w''\in T'} e^{c_{\widetilde{F}}(w,\,w'')}\,d\mu_{W^{su}(w'')}(w)\,d\nu_{T'}(w'')\,.$$

Hence the measures on the sets U_T defined by the right hand side of Equation (125) glue together to define a measure $\widetilde{m}^{\nu,\,\mu}$ on $T^1\widetilde{M}$. The uniqueness follows from the fact that the open sets U_T cover $T^1\widetilde{M}$.

The measure $\widetilde{m}^{\nu,\,\mu}$ is locally finite since ν_T and $\mu_{W^{su}(w')}$ are locally finite. It is Γ-invariant since the families ν and $(\mu_{W^{su}(w')})_{w'\in T^1\widetilde{M}}$ are Γ-equivariant and the cocycle $c_{\widetilde{F}}$

is Γ-invariant. The last claim of the lemma follows from the fact that ν_T gives full measure to the set of $w' \in T$ such that $w'_- \in \Lambda_c\Gamma$ and $\mu_{W^{su}(w')}$ to the elements $w \in W^{su}(w')$ such that $w_+ \in \Lambda_c\Gamma$. $\square$

Note that the measure $m^{\nu,\,\mu}$ is not necessarily finite (see for instance [Bur]). We will turn it into a probability measure by multiplying it by a convenient bump function.

We fix a nonnegative (continuous) $\rho \in \mathscr{C}_c(T^1M;\mathbb{R})$ with large enough (compact) support, so that

$$\int_{T^1M} \rho\, dm_F > 0 \quad \text{and} \quad \int_{T^1M} \rho\, dm^{\nu,\,\mu} = 1\ .$$

With $p : T^1\widetilde{M} \to T^1M$ the canonical projection, let $\widetilde{\rho} = \rho \circ p$ be the lift of ρ to $T^1\widetilde{M}$. Consider the measure $\widetilde{\Pi} = \widetilde{\rho}\, \widetilde{m}^{\nu,\,\mu}$ on $T^1\widetilde{M}$, which induces a probability measure

$$\Pi = \rho\, m^{\nu,\,\mu}$$

on T^1M.

Step 2. In this step, we construct two families $(I_r)_{r>0}$ and $(J_r)_{r>0}$ of diffusion operators along the strong unstable leaves, acting on the locally integrable functions on T^1M, and prove an adjointness property for them which will be crucial for the proof of Theorem 10.4.

Recall (see Subsection 2.4) that $d_{W^{\rm su}(v)}$ is the Hamenstädt distance on the strong unstable leaf of $v \in T^1\widetilde{M}$. We will denote by $B^{su}(v, r)$ the open ball of center v and radius r for the Hamenstädt distance $d_{W^{su}(v)}$ on $W^{\rm su}(v)$. (It is in general different from the open ball of center v and radius r for the induced Riemannian distance, that was denoted in the same way in Chapter 6: this will not cause any problem since these Riemannian balls are only used in Chapter 6). The distances $d_{W^{\rm su}(v)}$ and balls $B^{su}(v, r)$ were denoted by $d_{H_+(v)}$ and $B^+(v, r)$ in [Rob1] and [Sch3]. For all $v \in T^1\widetilde{M}$, $r > 0$ $\gamma \in \Gamma$ and $t \in \mathbb{R}$, by Equation (8), we have

$$\gamma B^{su}(v, r) = B^{su}(\gamma v, r)\ ,$$

and

$$\phi_t\big(B^{su}(v, r)\big) = B^{su}(\phi_t v, e^t r)\ . \tag{126}$$

Similarly, the open balls $B^{ss}(v, r)$ of center v and radius r for the Hamenstädt distance $d_{W^{ss}(v)}$ on the strong stable leaves $W^{\rm ss}(v)$ satisfy $\gamma B^{ss}(v, r) = B^{ss}(\gamma v, r)$ and

$$\phi_t\big(B^{ss}(v, r)\big) = B^{ss}(\phi_t v, e^{-t} r)\ . \tag{127}$$

Recall that $p : T^1\widetilde{M} \to T^1M$ is the canonical projection. For every $r > 0$ and for all measurable nonnegative maps $\psi : T^1M \to \mathbb{R}$, let $\widetilde{\psi} = \psi \circ p$ be the *lift* of ψ to $T^1\widetilde{M}$. Let us define $\widetilde{I_r}\widetilde{\psi} : T^1\widetilde{M} \to \mathbb{R}$ as an integral of $\widetilde{\psi}$ on the balls of radius r of the strong unstable leaves:

$$\forall\, u \in T^1\widetilde{M}, \quad \widetilde{I_r}\widetilde{\psi}\,(u) = \int_{v \in B^{su}(u,\, r)} \widetilde{\psi}(v)\, e^{c_{\widetilde{F}}(v,\, u)}\, d\mu_{W^{su}(u)}(v)\ . \tag{128}$$

We define similarly

$$\forall\, u \in T^1\widetilde{M}, \quad \widetilde{J_r}\widetilde{\psi}\,(u) = \int_{v \in B^{su}(u,\, r)} \widetilde{\psi}(v)\, d\mu_{W^{su}(u)}(v)\ . \tag{129}$$

The maps $\widetilde{I}_r\widetilde{\psi}$ and $\widetilde{J}_r\widetilde{\psi}$ are Γ-invariant, by Equation (43) and the Γ-invariance of $c_{\widetilde{F}}$ and $\widetilde{\psi}$. Hence they define measurable nonnegative maps $I_r\psi, J_r\psi : T^1M \to [0, +\infty]$ whose lifts are $\widetilde{I}_r\widetilde{\psi}, \widetilde{J}_r\widetilde{\psi}$. Note that $I_r\psi = J_r\psi$ if $F = 0$.

Since $\widetilde{\psi}$ is nonnegative, then for every $u \in T^1\widetilde{M}$, the maps $r \mapsto \widetilde{I}_r\widetilde{\psi}\,(u)$ and $r \mapsto \widetilde{J}_r\widetilde{\psi}\,(u)$ are nondecreasing. The operators $\widetilde{I}_r$, $\widetilde{J}_r$, I_r and J_r are positive: For instance, if $\psi \leq \psi'$ then $I_r\psi \leq I_r\psi'$.

If $\widetilde{\psi}$ is continuous and positive at an element of the support of $\mu_{W^{su}(u)}$ (which is the set of vectors $v \in W^{su}(u)$ such that $v_+ \in \Lambda\Gamma$), then $\widetilde{I}_r\widetilde{\psi}\,(u) > 0$. In particular, we have $\widetilde{I}_r\widetilde{\rho}\,(u) > 0$ if $\widetilde{\rho}(u) > 0$ and $u_+ \in \Lambda\Gamma$. Hence the Γ-invariant map $\frac{\widetilde{\rho}}{\widetilde{I}_r\widetilde{\rho}}$, defined to have value 0 at $u \in T^1\widetilde{M}$ if $\widetilde{\rho}(u) = 0$ or $u_+ \notin \Lambda\Gamma$, induces a measurable nonnegative map $\frac{\rho}{I_r\rho}$ on T^1M.

Let us denote by 1 the constant map with value 1 on T^1M and on $T^1\widetilde{M}$, which satisfies $\widetilde{I}_r 1(u) > 0$ if $u_+ \in \Lambda\Gamma$. We will apply the mixing property of Theorem 8.2 when B is a Hamenstädt ball $B^{su}(u, r)$, giving that, for every $u \in T^1M$ and every nonnegative $\psi \in \mathscr{C}_c(T^1M; \mathbb{R})$,

$$\lim_{t \to +\infty} I_r(\psi \circ \phi_t)(u) = \frac{I_r 1\,(u)}{\|m_F\|} \int_{T^1M} \psi \, dm_F \; . \tag{130}$$

The diffusion operators I_r on the strong unstable balls have the following commutation property with the geodesic flow.

Lemma 10.8 *For all $t \in \mathbb{R}$, $r > 0$, $u \in T^1M$ and $\psi : T^1M \to \mathbb{R}$ measurable nonnegative, we have*

$$I_{re^t}(\psi \circ \phi_{-t})(\phi_t u) = e^{t\delta - \int_0^t F(\phi_s u)\, ds}\; I_r\psi\,(u) \; .$$

Proof. By the equations (45), (19) and (21), for all $t \in \mathbb{R}$, $u \in T^1\widetilde{M}$ and $w \in W^{su}(u)$, we have

$$\begin{aligned} d\,\mu_{W^{su}(\phi_t u)}(\phi_t w) &= e^{-C_{F-\delta,\, w_+}(\pi(\phi_t w),\, \pi(w))}\; d\,\mu_{W^{su}(u)}(w) \\ &= e^{-C_{F\circ\iota-\delta,\, w_-}(\pi(w),\, \pi(\phi_t w))}\; d\,\mu_{W^{su}(u)}(w) \; . \end{aligned}$$

Hence for all $t \in \mathbb{R}$, $r > 0$, $u \in T^1\widetilde{M}$ and $\widetilde{\psi} : T^1M \to \mathbb{R}$ measurable nonnegative, by the definition of the operators $\widetilde{I}_r$ and by Equation (126), using the change of variables $v = \phi_t w$ and the definition of the cocycle $c_{\widetilde{F}}$ in Equation (119), we have

$$\begin{aligned} \widetilde{I}_{re^t}(\widetilde{\psi} \circ \phi_{-t})(\phi_t u) &= \int_{v \in B^{su}(\phi_t u,\, re^t)} \widetilde{\psi}(\phi_{-t} v)\; e^{c_{\widetilde{F}}(v,\, \phi_t u)}\; d\mu_{W^{su}(\phi_t u)}(v) \\ &= \int_{w \in B^{su}(u,\, r)} \widetilde{\psi}(w)\; e^{c_{\widetilde{F}}(\phi_t w,\, \phi_t u)} e^{-C_{F\circ\iota-\delta,\, w_-}(\pi(w),\, \pi(\phi_t w))}\; d\,\mu_{W^{su}(u)}(w) \\ &= e^{-C_{F\circ\iota-\delta,\, u_-}(\pi(u),\, \pi(\phi_t u))}\; \widetilde{I}_r\widetilde{\psi}(u) = e^{-\int_0^t (\widetilde{F}(\phi_s u) - \delta)\, ds}\; \widetilde{I}_r\widetilde{\psi}(u) \; . \end{aligned} \tag{131}$$

The result follows. $\square$

The operators I_r and J_r satisfy the following crucial adjointness property for the scalar product of $\mathbb{L}^2(m^{\nu,\mu})$.

Lemma 10.9 *For all $r > 0$ and $\psi, \varphi : T^1M \to \mathbb{R}$ measurable nonnegative, we have*

$$\int_{T^1M} I_r\psi\ \varphi\ dm^{\nu,\,\mu} = \int_{T^1M} \psi\ J_r\varphi\ dm^{\nu,\,\mu}\ .$$

Proof. We assume that Γ is torsion free: the general case follows by proving, for every $k \in \mathbb{N} - \{0\}$, an analogous statement for the restriction of $m^{\nu,\,\mu}$ to the image in T^1M of the Γ-invariant Borel subset of points in $T^1\widetilde{M}$ whose stabiliser in Γ has order k, and by summation (see Subsection 2.6).

By [Rob1, §1C], there exists hence a weak fundamental domain in $T^1\widetilde{M}$ with $\widetilde{m}^{\nu,\,\mu}$-negligible boundary, that is an open subset $\mathscr{D}$ in $T^1\widetilde{M}$ such that $\gamma\mathscr{D} \cap \mathscr{D}$ is empty for every $\gamma \in \Gamma - \{\mathrm{id}\}$ and $\bigcup_{\gamma\in\Gamma} \gamma\,\overline{\mathscr{D}} = T^1\widetilde{M}$, and such that $\widetilde{m}^{\nu,\,\mu}(\partial\mathscr{D}) = 0$.

Let $T = W^s(v)$ be a stable leaf in $T^1\widetilde{M}$ such that v_+ is not an atom of μ_x. In particular, $U_T = \{w \in T^1\widetilde{M}\ :\ w_- \neq v_+\}$ has full measure for $\widetilde{m}^{\nu,\,\mu}$. Recall that $\mathbb{1}_A$ is the characteristic function of a subset A. Using

- Lemma 10.7 and Equation (128) for the second equality,
- the equality $W^{su}(w) = W^{su}(w')$ and the cocycle property of $c_{\widetilde{F}}$ for the third one,
- an elementary argument using Γ and $\mathscr{D}$, the fact that $w \in B^{su}(u,r)$ if and only if $u \in B^{su}(w,r)$ and Fubini's theorem for positive functions for the fourth one, we have

$$\begin{aligned}
\int_{T^1M} I_r\psi\ \varphi\ dm^{\nu,\,\mu} &= \int_{T^1\widetilde{M}} \widetilde{I_r\psi}\ \ \widetilde{\varphi}\ \mathbb{1}_{\mathscr{D}}\ d\widetilde{m}^{\nu,\,\mu}\\
&= \int_{w'\in T} \int_{w\in W^{su}(w')} \Big(\int_{u\in B^{su}(w,\,r)} \widetilde{\psi}(u)\ e^{c_{\widetilde{F}}(u,\,w)}\ d\mu_{W^{su}(w)}(u)\Big)\\
&\qquad\qquad \widetilde{\varphi}(w)\ \mathbb{1}_{\mathscr{D}}(w)\ e^{c_{\widetilde{F}}(w,\,w')}\ d\mu_{W^{su}(w')}(w)\ d\nu_T(w')\\
&= \int_{w'\in T} \int_{w\in W^{su}(w')} \int_{u\in W^{su}(w')} \widetilde{\varphi}(w)\ \widetilde{\psi}(u)\ \mathbb{1}_{\mathscr{D}}(w)\ \mathbb{1}_{B^{su}(w,\,r)}(u)\\
&\qquad\qquad e^{c_{\widetilde{F}}(u,\,w')}\ d\mu_{W^{su}(w')}(u)\ d\mu_{W^{su}(w')}(w)\ d\nu_T(w')\\
&= \int_{w'\in T} \int_{u\in W^{su}(w')} \int_{w\in W^{su}(w')} \widetilde{\varphi}(w)\ \widetilde{\psi}(u)\ \mathbb{1}_{\mathscr{D}}(u)\ \mathbb{1}_{B^{su}(u,\,r)}(w)\\
&\qquad\qquad e^{c_{\widetilde{F}}(u,\,w')}\ d\mu_{W^{su}(w')}(w)\ d\mu_{W^{su}(w')}(u)\ d\nu_T(w')\\
&= \int_{T^1M} \psi\ J_r\varphi\ dm^{\nu,\,\mu}\ ,
\end{aligned}$$

as required. $\square$

From now on, we consider a nonnegative $\psi \in \mathscr{C}_c(T^1M;\mathbb{R})$. For every $r > 0$, applying the above lemma with $\varphi = \frac{\rho}{I_r(\rho)}$, we have

$$\int_{T^1M} \frac{I_r\psi}{I_r\rho}\ \rho\ dm^{\nu,\,\mu} = \int_{T^1M} \psi\ J_r\big(\frac{\rho}{I_r\rho}\big)\ dm^{\nu,\,\mu}\ . \tag{132}$$

In the next three steps, we will prove that, up to positive multiplicative constants, the right hand side of this equality is bounded from above by $\int_{T^1M} \psi\ dm^{\nu,\,\mu}$, and that its left hand side is bounded from below by $\int_{T^1M} \psi\ dm_F$ if r is large enough. Since this is valid for any nonnegative $\psi \in \mathscr{C}_c(T^1M;\mathbb{R})$, this will imply that $m^{\nu,\,\mu}$ is absolutely continuous with respect to m_F.

Step 3. In this step, let us prove that the map $J_r(\frac{\rho}{I_r\rho})$ is bounded from above on a subset of T^1M of full measure with respect to $m^{\nu,\mu}$, which implies that the right hand side of Equation (132) is bounded from above by a constant times $\int_{w\in T^1M}\psi\; dm^{\nu,\mu}$.

Recalling that $\widetilde{M}$ has pinched negative curvature, the following result of polynomial growth of horospheres is due to Roblin [Rob1, Prop. 6.3 (a)] (see also [Bowd]). Recall that $\widetilde{\Omega}_c\Gamma$ is the set of $v\in T^1\widetilde{M}$ such that $v_-, v_+\in\Lambda_c\Gamma$, and that $\Omega_c\Gamma$ is its image in T^1M.

Lemma 10.10 (Roblin) *There exists $N\in\mathbb{N}-\{0\}$ such that for all $w\in\widetilde{\Omega}_c\Gamma$ and $r>0$, there exist $w_1,\dots,w_N\in W^{su}(w)$ such that*

$$B^{su}(w,r)\;\subset\;\bigcup_{i=1}^{N}\;B^{su}\Big(w_i,\frac{r}{2}\Big)\;.$$

Lemma 10.11 *For all $r>0$ and $w\in\Omega_c\Gamma$, we have*

$$J_r\Big(\frac{\rho}{I_r\rho}\Big)(w)\le N\;.$$

Proof. Let us fix $r>0$ and $w\in\widetilde{\Omega}_c\Gamma$, and let $w_1,\dots,w_N\in T^1\widetilde{M}$ be as in Lemma 10.10. For all $i\in\{1,\dots,N\}$ and $u\in B^{su}(w_i,\frac{r}{2})$, since $B^{su}(u,r)$ then contains $B^{su}(w_i,\frac{r}{2})$, we have

$$\begin{aligned}\widetilde{I}_r\widetilde{\rho}\,(u)&=\int_{v\in B^{su}(u,\,r)}\widetilde{\rho}\,(v)\;e^{c_{\widetilde{F}}(v,\,u)}\;d\mu_{W^{su}(u)}(v)\\&\ge e^{c_{\widetilde{F}}(w,\,u)}\int_{v\in B^{su}(w_i,\,\frac{r}{2})}\widetilde{\rho}\,(v)\;e^{c_{\widetilde{F}}(v,\,w)}\;d\mu_{W^{su}(w)}(v)\;.\end{aligned}$$

Hence

$$\begin{aligned}&\int_{u\in B^{su}(w_i,\,\frac{r}{2})}\frac{\widetilde{\rho}\,(u)}{\widetilde{I}_r\widetilde{\rho}\,(u)}\;d\mu_{W^{su}(w)}(u)\\\le&\int_{u\in B^{su}(w_i,\,\frac{r}{2})}\frac{\widetilde{\rho}\,(u)\;e^{c_{\widetilde{F}}(u,\,w)}}{\int_{v\in B^{su}(w_i,\,\frac{r}{2})}\widetilde{\rho}\,(v)\;e^{c_{\widetilde{F}}(v,\,w)}\;d\mu_{W^{su}(w)}(v)}\;d\mu_{W^{su}(w)}(u)=1\;.\end{aligned}$$

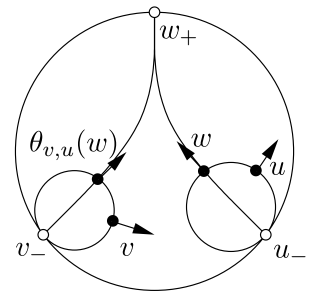
Using Lemma 10.10, the result follows. □

As required, since $\Omega_c\Gamma$ has full $m^{\nu,\mu}$-measure by Lemma 10.7, this lemma implies that

$$\int_{T^1M}\psi\;J_r\big(\frac{\rho}{I_r\rho}\big)\;dm^{\nu,\mu}\le N\int_{T^1M}\psi\;dm^{\nu,\mu}\;. \tag{133}$$

Step 4. In this most technical step, after giving the useful notation, we prove some continuity properties of the diffusion operators $I_r\psi\,(u)$ on the strong unstable balls, as functions of u. For this, we will adapt the technical lemmas of [Rob1, Sect. 1H] to the presence of a possibly unbounded potential, and add some more lemmas using the Hölder regularity of F needed to control the Gibbs cycocles.

For all $u, v \in T^1\widetilde{M}$ such that $u_- \neq v_-$, recall (see Subsection 3.9 before Equation (46)) that the map $\theta_{v,\,u} = \theta^{su}_{v,\,u} : \{w \in W^{su}(u) \ : \ w_+ \neq v_-\} \rightarrow \{w \in W^{su}(v) \ : \ w_+ \neq u_-\}$ is the homeomorphism sending w to the unique element in $W^{su}(v) \cap W^s(w)$. Note that $\theta_{v',\,u'} = \theta_{v,\,u}$ if $u' \in W^{su}(u)$ and $v' \in W^{su}(v)$. We have $\theta_{u,\,v} = {\theta_{v,\,u}}^{-1}$ and, whenever defined, $\theta_{w,\,u} = \theta_{w,\,v} \circ \theta_{v,\,u}$. For every $t \in \mathbb{R}$, we have



$$\phi_t \circ \theta_{v,\,u} = \theta_{\phi_t v,\,u} \quad \text{and} \quad \theta_{v,\,u} \circ \phi_t = \theta_{v,\,\phi_{-t}u} \ . \tag{134}$$

We recall the definition of the canonical neighbourhoods (called *dynamical cells*) of the unit tangent vectors, adapted to variable curvature. For all $u \in T^1\widetilde{M}$ and $r_1, r_2, r_3 > 0$,

$$\mathscr{V}_{r_1,\,r_2,\,r_3}(u) = \bigcup_{|s|<r_3} \phi_s\Big(\bigcup_{v \in B^{ss}(u,\,r_1)} \theta_{v,\,u}(B^{su}(u, r_2))\Big) \ . \tag{135}$$

The sets $\mathscr{V}_{r_1,\,r_2,\,r_3}(u)$ are nondecreasing in r_1, r_2, r_3 and form, as r_1, r_2, r_3 range over $]0, +\infty[$, a neighbourhood basis of the vector u in $T^1\widetilde{M}$. Furthermore, for every $t \in \mathbb{R}$, by the equations (134), (126) and (127), we have

$$\phi_t\big(\mathscr{V}_{r_1,\,r_2,\,r_3}(u)\big) = \mathscr{V}_{e^{-t}r_1,\,e^t r_2,\,r_3}(\phi_t u) \ . \tag{136}$$

For every $u \in T^1\widetilde{M}$, let

$$u_\epsilon = \bigcup_{|s|<\epsilon} \phi_s B^{ss}(u, \epsilon) \ .$$

The sets u_ϵ are nondecreasing in ϵ and form, as ϵ ranges over $]0, 1]$, a neighbourhood basis of the vector u in its stable leaf $W^s(u)$. Furthermore, for all $\gamma \in \Gamma$ and $t \geq 0$, we have $\gamma(u_\epsilon) = (\gamma u)_\epsilon$ and, by Equation (127),

$$\phi_t(u_\epsilon) \subset (\phi_t u)_\epsilon \quad \text{and} \quad (\phi_{-t}u)_\epsilon \subset \phi_{-t}(u_\epsilon) \ . \tag{137}$$

The following result contains the elementary properties of the notation introduced above. It says that if v is in a small dynamical cell around u, then the change of strong unstable leaf map $\theta_{v,u}$ doesn't move points too much. The important feature of this lemma is the uniformity of the constants, which will be useful to control the variations of the potential function and of the Gibbs cocycle.

Lemma 10.12 *For every $\epsilon > 0$, if $r_1, r_2, r_3 > 0$ are small enough, then for every $u \in T^1\widetilde{M}$, we have*

(1) for all $v \in \mathscr{V}_{r_1,\,r_2,\,r_3}(u)$ and $w \in B^{su}(u, 2)$, we have

$$\theta_{v,u}(w) \in w_\epsilon \ ,$$

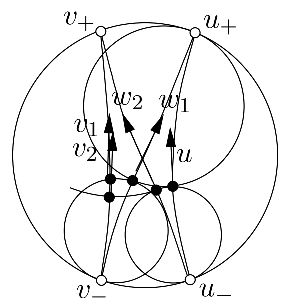
(2) for all $r \in [\frac{1}{2}, 2]$ and $v \in \mathscr{V}_{r_1,\,r_2,\,r_3}(u)$, we have

$$B^{su}(v, r\, e^{-\epsilon}) \subset \theta_{v,u}(B^{su}(u, r)) \subset B^{su}(v, re^{\epsilon}) \ .$$

Proof. (1) Let us recall the statement [Rob1, Lem. 1.13]: for every $\epsilon' > 0$, there exists $r(\epsilon') > 0$ such that for all $u' \in T^1\widetilde{M}$, $v' \in B^{ss}(u', r(\epsilon'))$ and $w' \in B^{su}(u', 3)$ (the statement is given in loc. cit. with the number 2 instead of this number 3, but the proof is the same), we have

$$\theta_{v',u'}(w') \in (w')_{\epsilon'}\ .$$

Now, for every $\epsilon > 0$, let $\epsilon' = \min\{r(\frac{\epsilon}{3}), \frac{\epsilon}{3}\} > 0$. Let $r_1 \in\,]0, r(\epsilon')]$, $r_2 \in\,]0,1]$ and $r_3 \in\,]0, \frac{\epsilon}{3}]$.



Let $u \in T^1\widetilde{M}$, $w \in B^{su}(u,2)$ and $v \in \mathscr{V}_{r_1,r_2,r_3}(u)$. By the definition of $\mathscr{V}_{r_1,r_2,r_3}(u)$, there exist $s_1 \in\,]-r_3, r_3[$, $w_1 \in B^{ss}(u, r_1)$ and $w_2 \in B^{su}(u, r_2)$ such that if $v_1 = \theta_{w_1,u}(w_2)$, then $v = \phi_{s_1} v_1$. By [Rob1, Lem. 1.13] as recalled above, since $r_2 \leq 3$ and $r_1 \leq r(\epsilon')$, there exist $s_2 \in\,]-\epsilon', \epsilon'[$ and $v_2 \in B^{ss}(w_2, \epsilon')$ such that $v_1 = \phi_{s_2} v_2$, so that in particular $v = \phi_{s_1+s_2} v_2$.

By the triangle inequality, we have $w \in B^{su}(u,2) \subset B^{su}(w_2, 2+r_2) \subset B^{su}(w_2,3)$. Since $W^{su}(w) = W^{su}(u) = W^{su}(w_2)$ and $v_2 \in B^{ss}(w_2, r(\frac{\epsilon}{3}))$, applying again [Rob1, Lem. 1.13] as recalled above, there exist $s_3 \in\,]-\frac{\epsilon}{3}, \frac{\epsilon}{3}[$ and $v_3 \in B^{ss}(w, \frac{\epsilon}{3})$ such that $\theta_{v_2,u}(w) = \theta_{v_2,w_2}(w) = \phi_{s_3} v_3$. Hence we have by Equation (134) that

$$\theta_{v,\,u}(w) = \theta_{\phi_{s_1} v_1,\,u}(w) = \theta_{\phi_{s_1+s_2} v_2,\,u}(w) = \phi_{s_1+s_2}\theta_{v_2,\,u}(w) = \phi_{s_1+s_2+s_3} v_3 \in w_\epsilon\ .$$

(2) By [Rob1, Coro. 1.14], for every $\epsilon > 0$, there exist $r'_1, r'_2, r'_3 > 0$ such that for all $u' \in T^1\widetilde{M}$ and $v' \in \mathscr{V}_{r'_1,r'_2,r'_3}(u')$, we have

$$B^{su}(v', e^{-\epsilon}) \subset \theta_{v',u'}(B^{su}(u',1)) \subset B^{su}(v', e^{\epsilon})\ . \tag{138}$$

Assume that $r_1 \in\,]0, \frac{r'_1}{2}]$, $r_2 \in\,]0, \frac{r'_2}{2}]$ and $r_3 \leq r'_3$. For every $r \in [\frac{1}{2}, 2]$, let $t = \log r$. For every $v \in \mathscr{V}_{r_1,r_2,r_3}(u)$, let $u' = \phi_{-t} u$ and $v' = \phi_{-t} v$. Then

$$v' \in \phi_{-t}\mathscr{V}_{r_1,\,r_2,\,r_3}(u) = \mathscr{V}_{e^t r_1,\,e^{-t} r_2,\,r_3}(u') \subset \mathscr{V}_{r'_1,\,r'_2,\,r'_3}(u')\ ,$$

and the result follows by applying the map ϕ_t to the inclusions (138), while using the equations (126) and (134). □

We will need the following consequence of the Hölder regularity of $\widetilde{F}$ (which is empty if $\widetilde{F} = 0$).

Lemma 10.13 *For every $\epsilon > 0$, if $\epsilon' > 0$ is small enough, then for all $w \in T^1\widetilde{M}$ and $w' \in w_{\epsilon'}$, we have*

$$|\, C_{F-\delta,\,w_+}(\pi(w), \pi(w'))\,| \leq \epsilon \Big(1 + \sup_{\pi^{-1}(B(\pi(w),\,1))} |\widetilde{F}|\Big)\ .$$

Proof. By the Hölder-continuity of $\widetilde{F}$, there exist two constants $c > 0$ and $\alpha \in]0,1]$ such that for all u, v in $T^1\widetilde{M}$ with $d(u,v) \leq 1$, we have $|\,\widetilde{F}(u) - \widetilde{F}(v)\,| \leq c\, d(u,v)^\alpha$. For all

$\epsilon' > 0$, $w \in T^1\widetilde{M}$ and $w' \in w_{\epsilon'}$, there exist $s \in \,]-\epsilon', \epsilon'[$ and $w'' \in B^{ss}(w, \epsilon')$ such that $w' = \phi_s w''$, by the definition of $w_{\epsilon'}$.

By Equation (9) (exchanging the strong unstable and the strong stable foliation), for every $t \in [0, s]$, we have

$$\begin{aligned} d(\pi(w), \pi(\phi_t w')) &\leq d(\pi(w), \pi(w'')) + d(\pi(w''), \pi(\phi_t w')) \\ &\leq d_{W^{ss}(w)}(w, w'') + |s + t| \leq 3\epsilon' \;. \end{aligned}$$

By Equation (9) and Equation (11), there exists a constant $c' > 0$ such that, for every $t \geq 0$,

$$d(\phi_t w, \phi_t w'') \leq c' d_{W^{ss}(\phi_t w)}(\phi_t w, \phi_t w'') \leq c' e^{-t} d_{W^{ss}(w)}(w, w'') \leq c' \epsilon' e^{-t} \leq 1 \;,$$

if ϵ' is small enough. Therefore

$$\begin{aligned} \big|\, C_{F-\delta,\, w_+}(\pi(w), \pi(w')) \,\big| &= \big|\, C_{F-\delta,\, w_+}(\pi(w), \pi(w'')) + C_{F-\delta,\, w_+}(\pi(w''), \pi(w')) \,\big| \\ &\leq \Big| \int_0^{+\infty} \widetilde{F}(\phi_t w) - \widetilde{F}(\phi_t w'') \; dt \,\Big| + \Big| \int_0^{s} \big(\widetilde{F}(\phi_t w') - \delta\big) \; dt \,\Big| \\ &\leq \frac{c}{\alpha}(c'\epsilon')^{\alpha} + \epsilon' \Big(\max_{\pi^{-1}(B(\pi(w),\, 3\epsilon'))} |\widetilde{F}| + \delta \Big) \;. \end{aligned}$$

This proves the result. $\square$

Remark. Some version of this result may also be deduced from Lemma 3.4, as follows.

First observe that

$$\forall\; v \in u_{\epsilon}, \quad d(\pi(v), \pi(u)) < 2\epsilon \;. \tag{139}$$

Indeed, if $v = \phi_s w$ where $|s| < \epsilon$ and $w \in B^{ss}(u, \epsilon)$, then $d(\pi(v), \pi(w)) = |s| < \epsilon$ and by Equation (9) (exchanging stable and unstable foliations), we have

$$d(\pi(u), \pi(w)) \leq d_{W^{ss}(u)}(u, w) < \epsilon \;,$$

so that Equation (139) holds by the triangle inequality.

Now by Equation (139) and by Lemma 3.4, there exist two constants c_1, c_2 such that for every $w' \in w_{\epsilon'}$,

$$|\, C_{F-\delta,\, w_+}(\pi(w), \pi(w')) \,| \leq c_1 (2\epsilon')^{c_2} + (2\epsilon') \Big(\max_{\pi^{-1}(B(\pi(w),\, 2\epsilon'))} |\widetilde{F}| + \delta \Big) \;,$$

which proves the result.

A second consequence of the Hölder regularity of $\widetilde{F}$ that we will need is the following result (again empty if $F = 0$).

Lemma 10.14 *For all $\epsilon > 0$ and $T \geq 0$, if r_1, r_2, $r_3 > 0$ are small enough, then for all $u \in T^1\widetilde{M}$ and $v \in \mathscr{V}_{r_1, r_2, r_3}(u)$, we have*

$$\Big| \int_0^{T} \widetilde{F}(\phi_t u) - \widetilde{F}(\phi_t v) \; dt \,\Big| \leq \epsilon \;.$$

Proof. Let $c > 0$ and $\alpha \in \,]0, 1]$ be as in the beginning of the proof of Lemma 10.13. As seen in the proof of Lemma 10.12 (1), for every $\epsilon' > 0$, if r_1, r_2, r_3 are small enough, then for all $u \in T^1\widetilde{M}$ and $v \in \mathscr{V}_{r_1,r_2,r_3}(u)$, there exist $w_2 \in B^{su}(u, r_2)$, $v_2 \in B^{ss}(w_2, \epsilon')$ and $s \in \,]-\epsilon', \epsilon'[$ such that $v = \phi_s v_2$. Let $t \in [0, T]$.

By Equation (9) and Equation (8), there exists a constant $c' > 0$ such that

$$d(\phi_t u, \phi_t w_2) \leq c'\, d_{W^{\rm su}(\phi_t u)}(\phi_t u, \phi_t w_2) = c'\, e^t\, d_{W^{\rm su}(u)}(u, w_2) \leq c'\, e^T r_2 \leq 1$$

if r_2 is small enough. Similarly,

$$d(\phi_t w_2, \phi_t v_2) \leq c'\, d_{W^{\rm ss}(\phi_t w_2)}(\phi_t w_2, \phi_t v_2) = c'\, e^{-t}\, d_{W^{\rm ss}(w_2)}(w_2, v_2) \leq c'\, \epsilon' \leq 1$$

if ϵ' is small enough. We also have, if ϵ' is small enough,

$$d(\phi_{t+s} v, \phi_t v) = |s| \leq \epsilon' \leq 1\,.$$

Therefore, by the Hölder regularity of $\widetilde{F}$, we have

$$\begin{aligned}
&|\,\widetilde{F}(\phi_t u) - \widetilde{F}(\phi_t v)\,| \\
&\leq |\,\widetilde{F}(\phi_t u) - \widetilde{F}(\phi_t w_2)\,| + |\,\widetilde{F}(\phi_t w_2) - \widetilde{F}(\phi_t v_2)\,| + |\,\widetilde{F}(\phi_t v_2) - \widetilde{F}(\phi_t v)\,| \\
&\leq c\,(c'\, e^T r_2)^\alpha + c\,(c'\, \epsilon')^\alpha + c\,\epsilon'^\alpha\,.
\end{aligned}$$

By integrating over $[0, T]$, this proves the result, if ϵ' and r_2 are chosen small enough. $\Box$

The following lemma (which is still empty if $F = 0$), giving a uniform continuity property of the cocycle $c_{\widetilde{F}}$, will be useful to prove the continuity properties of $u \mapsto \widetilde{I}_r\widetilde{\varphi}(u)$.

Lemma 10.15 *For every $\epsilon > 0$, if $\epsilon', r'_1, r'_2, r'_3 > 0$ are small enough, then for all $u \in T^1\widetilde{M}$, $w \in B^{su}(u, 2)$, $v \in \mathscr{V}_{r'_1,r'_2,r'_3}(u)$ and $w' \in w_{\epsilon'} \cap W^{su}(v)$, we have*

$$|\,c_{\widetilde{F}}(w',\, v) - c_{\widetilde{F}}(w,\, u)\,| \leq \epsilon\,.$$

A less precise version of this result follows quite directly from the Hölder-continuity of $c_{\widetilde{F}}$. The proof shows that this result also holds if $w' \in \mathscr{V}_{r'_1,r'_2,r'_3}(u) \cap W^{su}(v)$. But we will use the explicit form given above (with $\frac{\epsilon}{2}$ instead of ϵ !).

Proof. Let us fix $\epsilon > 0$. Let $c > 0$ and $\alpha \in \,]0, 1]$ be as in the beginning of the proof of Lemma 10.13. By Equation (9) and Equation (8), there exists a constant $c' > 0$ such that, for all $t \geq 0$, $u' \in T^1\widetilde{M}$ and $u'' \in B^{su}(u', 3)$, we have

$$d(\phi_{-t} u', \phi_{-t} u'') \leq c'\, d_{W^{su}(\phi_{-t} u')}(\phi_{-t} u', \phi_{-t} u'') \leq c'\, e^{-t}\, d_{W^{su}(u')}(u', u'') \leq 3\, c'\, e^{-t}\,.$$

Let $T = \max\{0, -\log(\frac{1}{3c'} \sqrt[\alpha]{\frac{\epsilon\alpha}{4c}}\}$. By the Hölder regularity of $\widetilde{F}$, for every $w \in B^{su}(u, 2) \subset B^{su}(u, 3)$, we have

$$\Big|\int_T^{+\infty} \widetilde{F}(\phi_{-t} w) - \widetilde{F}(\phi_{-t} u)\; dt\Big| \leq \int_T^{+\infty} c\,(3\, c'\, e^{-t})^\alpha = \frac{c}{\alpha}\,(3\, c'\, e^{-T})^\alpha \leq \frac{\epsilon}{4}\,.$$

If $\epsilon', r'_1, r'_2, r'_3$ are small enough, then for all $v \in \mathscr{V}_{r'_1,r'_2,r'_3}(u)$ and $w' \in w_{\epsilon'} \cap W^{su}(v)$, we have $w' \in B^{su}(v, 3)$ since $w \in B^{su}(u, 2)$, hence similarly

$$\Big|\int_T^{+\infty} \widetilde{F}(\phi_{-t} w') - \widetilde{F}(\phi_{-t} v)\; dt\Big| \leq \frac{\epsilon}{4}\,.$$

Since $w' \in w_{\epsilon'}$, there exist $s' \in \,]-\epsilon', \epsilon'[$ and $w'' \in B^{ss}(w, \epsilon')$ such that

$$w' = \phi_{s'} w'' = \phi_{s'}\, \theta_{w'',\, w}(w) ,$$

this last equality holding since $w'' \in W^s(w)$. Since $w \in B^{su}(w, \epsilon')$, we hence have

$$w' \in \mathscr{V}_{\epsilon',\, \epsilon',\, \epsilon'}(w) .$$

Let $r_1, r_2, r_3 > 0$ be the constants associated by Lemma 10.14 to $\frac{\epsilon}{4}$ and T. If ϵ', r_1', r_2', $r_3' > 0$ are small enough, then $\phi_{-T} v \in \mathscr{V}_{r_1, r_2, r_3}(\phi_{-T} u)$ and $\phi_{-T} w' \in \mathscr{V}_{r_1, r_2, r_3}(\phi_{-T} w)$ by Equation (136). Hence by Lemma 10.14, using a change of variable $t' = T - t$, we have

$$\Big| \int_0^T \widetilde{F}(\phi_{-t} u) - \widetilde{F}(\phi_{-t} v)\; dt \Big| = \Big| \int_0^T \widetilde{F}(\phi_{t'}(\phi_{-T} u)) - \widetilde{F}(\phi_{t'}(\phi_{-T} v))\; dt' \Big| \leq \frac{\epsilon}{4}$$

and similarly

$$\Big| \int_0^T \widetilde{F}(\phi_{-t} w) - \widetilde{F}(\phi_{-t} w')\; dt \Big| \leq \frac{\epsilon}{4} .$$

Now, by the definition of the cocycle $c_{\widetilde{F}}$, we have

$$\begin{aligned}
&|\, c_{\widetilde{F}}(w',\, v) - c_{\widetilde{F}}(w,\, u)\,| \\
&= \Big| \int_0^{+\infty} \widetilde{F}(\phi_{-t} w') - \widetilde{F}(\phi_{-t} v) - \widetilde{F}(\phi_{-t} w) + \widetilde{F}(\phi_{-t} u)\; dt \Big| \\
&\leq \Big| \int_T^{+\infty} \widetilde{F}(\phi_{-t} w') - \widetilde{F}(\phi_{-t} v)\; dt \Big| + \Big| \int_T^{+\infty} \widetilde{F}(\phi_{-t} w) - \widetilde{F}(\phi_{-t} u)\; dt \Big| \\
&\quad + \Big| \int_0^T \widetilde{F}(\phi_{-t} u) - \widetilde{F}(\phi_{-t} v)\; dt \Big| + \Big| \int_0^T \widetilde{F}(\phi_{-t} w) - \widetilde{F}(\phi_{-t} w')\; dt \Big| \\
&\leq \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{4} = \epsilon
\end{aligned}$$

This proves the result. $\square$

Let us now introduce the functional version of the notation u_ϵ. For every $\epsilon > 0$ and every nonnegative measurable map $\varphi : T^1 M \to \mathbb{R}$, whose lift to $T^1 \widetilde{M}$ is denoted by $\widetilde{\varphi}$, define

$$\widetilde{\varphi}_{\overline{\epsilon}} : u \mapsto \sup_{v \in u_\epsilon} \widetilde{\varphi}(v) \quad \text{and} \quad \widetilde{\varphi}_{\underline{\epsilon}} : u \mapsto \inf_{v \in u_\epsilon} \widetilde{\varphi}(v) .$$

Note that $\widetilde{\varphi}_{\overline{\epsilon'}} \leq \widetilde{\varphi}_{\overline{\epsilon}}$ and $\widetilde{\varphi}_{\underline{\epsilon'}} \geq \widetilde{\varphi}_{\underline{\epsilon}}$ if $\epsilon' \leq \epsilon$. The maps $\widetilde{\varphi}_{\overline{\epsilon}}$ and $\widetilde{\varphi}_{\underline{\epsilon}}$ are Γ-invariant, hence define nonnegative measurable maps $\varphi_{\overline{\epsilon}}$ and $\varphi_{\underline{\epsilon}}$ from $T^1 M$ to $\mathbb{R}$. For every $t \geq 0$, by Equation (137), we have

$$(\varphi \circ \phi_t)_{\overline{\epsilon}} \leq \varphi_{\overline{\epsilon}} \circ \phi_t \quad \text{and} \quad (\varphi \circ \phi_t)_{\underline{\epsilon}} \geq \varphi_{\underline{\epsilon}} \circ \phi_t . \tag{140}$$

The following result is the regularity property in u of $\widetilde{I}_r \widetilde{\varphi}\,(u)$ that will be needed in the next steps.

Proposition 10.16 *Let K be a compact subset of $T^1 \widetilde{M}$. For all $\epsilon > 0$ and $r \geq 1$, if $r_1, r_2, r_3 > 0$ are small enough, then for all $u \in \Gamma K$ and $v \in \mathscr{V}_{r_1, r_2, r_3}(u)$ and for every nonnegative measurable map $\varphi : T^1 M \to \mathbb{R}$, we have*

$$e^{-\epsilon} \widetilde{I}_{re^{-\epsilon}}(\widetilde{\varphi}_{\underline{\epsilon}})\,(u) \leq \widetilde{I}_r \widetilde{\varphi}\,(v) \leq e^{\epsilon} \widetilde{I}_{re^{\epsilon}}(\widetilde{\varphi}_{\overline{\epsilon}})\,(u) . \tag{141}$$

Proof. Let us first prove that we may assume without loss of generality that $r = 1$.

Assume that the result is true for $r = 1$. Let $\epsilon > 0$, $r \geq 1$ and $t = -\log r \leq 0$. For all $r_1, r_2, r_3 > 0$, $u \in T^1\widetilde{M}$ and $v \in \mathscr{V}_{e^t r_1,\, e^{-t} r_2,\, r_3}(u)$, we have $\phi_t v \in \mathscr{V}_{r_1,\, r_2,\, r_3}(\phi_t u)$ by Equation (136). Hence by the case $r = 1$ of this proposition applied to $\phi_t K$, $\frac{\epsilon}{2}$ and $\widetilde{\varphi} \circ \phi_{-t}$, we have, if r_1, r_2, r_3 are small enough,

$$e^{-\frac{\epsilon}{2}}\ \widetilde{I}_{e^{-\frac{\epsilon}{2}}}((\widetilde{\varphi} \circ \phi_{-t})_{\underline{\frac{\epsilon}{2}}})\ (\phi_t u) \leq \widetilde{I}_1(\widetilde{\varphi} \circ \phi_{-t})\ (\phi_t v) \leq e^{\frac{\epsilon}{2}}\ \widetilde{I}_{e^{\frac{\epsilon}{2}}}((\widetilde{\varphi} \circ \phi_{-t})_{\overline{\frac{\epsilon}{2}}})\ (\phi_t u)\ .$$

By the monotonicity properties of the maps $s \mapsto \widetilde{I}_s \widetilde{\varphi}'(u')$, $\epsilon' \mapsto \widetilde{\varphi}'_{\overline{\epsilon'}}(u')$ and $\epsilon' \mapsto \widetilde{\varphi}'_{\underline{\epsilon'}}(u')$ for every $u' \in T^1\widetilde{M}$ when $\varphi' : T^1\widetilde{M} \to \mathbb{R}$ is nonnegative, and by Equation (140) since $-t \geq 0$, we have

$$e^{-\frac{\epsilon}{2}}\ \widetilde{I}_{e^{-\epsilon}}(\widetilde{\varphi}_{\underline{\epsilon}} \circ \phi_{-t})\ (\phi_t u) \leq \widetilde{I}_1(\widetilde{\varphi} \circ \phi_{-t})\ (\phi_t v) \leq e^{\frac{\epsilon}{2}}\ \widetilde{I}_{e^{\epsilon}}(\widetilde{\varphi}_{\overline{\epsilon}} \circ \phi_{-t})\ (\phi_t u)\ .$$

By Lemma 10.8, we have

$$\begin{aligned} &e^{-\frac{\epsilon}{2}}\ e^{\delta t - \int_0^t \widetilde{F}(\phi_s u)\, ds}\ \widetilde{I}_{r\, e^{-\epsilon}}(\widetilde{\varphi}_{\underline{\epsilon}})\ (u) \\ &\qquad\qquad \leq e^{\delta t - \int_0^t \widetilde{F}(\phi_s v)\, ds}\ \widetilde{I}_r(\widetilde{\varphi})\ (v) \leq e^{\frac{\epsilon}{2}}\ e^{\delta t - \int_0^t \widetilde{F}(\phi_s u)\, ds}\ \widetilde{I}_{r\, e^{\epsilon}}(\widetilde{\varphi}_{\overline{\epsilon}})\ (u)\ . \end{aligned}$$

Dividing by $e^{\delta t}$ and applying Lemma 10.14 (with $\frac{\epsilon}{2}$ instead of ϵ), the general case of Proposition 10.16 follows.

Let us now prove Proposition 10.16 with $r = 1$. Let $\epsilon, r_1, r_2, r_3 > 0$, $u \in T^1\widetilde{M}$, $v \in \mathscr{V}_{r_1, r_2, r_3}(u)$ and let $\varphi : T^1M \to \mathbb{R}$ be a nonnegative measurable map. Since $e^{\epsilon}\widetilde{I}_{e^{\epsilon}}(\widetilde{\varphi}_{\overline{\epsilon}})\ (u)$ is nondecreasing in ϵ and since $e^{-\epsilon}\widetilde{I}_{e^{-\epsilon}}(\widetilde{\varphi}_{\underline{\epsilon}})\ (u)$ is nonincreasing in ϵ, we may assume that $\epsilon \leq \log 2$.

Let $\epsilon' \in \,]0, \epsilon[$ and $w \in B^{su}(u, e^{\pm\epsilon})$. By Lemma 10.12 (1), if $r_1, r_2, r_3 > 0$ are small enough (depending only on ϵ'), we have

$$\theta_{v,\,u}(w) \in w_{\epsilon'} \subset w_{\epsilon}\ . \tag{142}$$

By Equation (46), using the change of variables $w' = \theta_{v,\,u}(w)$, we have

$$\begin{aligned} &\int_{w' \in\, \theta_{v,\,u}(B^{su}(u,\, e^{\pm\epsilon}))} \widetilde{\varphi}(w')\ e^{c_{\widetilde{F}}(w',\, v)}\ d\mu_{W^{su}(v)}(w') \\ &= \int_{w \in B^{su}(u,\, e^{\pm\epsilon})} \widetilde{\varphi}(\theta_{v,u}(w))\ e^{c_{\widetilde{F}}(\theta_{v,u}(w),\, v) - c_{\widetilde{F}}(w,\, u) - C_{F-\delta,\, w_+}(\pi(\theta_{v,u}(w)),\, \pi(w))} \\ &\qquad\qquad\qquad\qquad\qquad\qquad e^{c_{\widetilde{F}}(w,\, u)}\ d\mu_{W^{su}(u)}(w)\ . \end{aligned} \tag{143}$$

By Lemma 10.12 (2) with respectively $r = e^{-\epsilon}$ and $r = e^{\epsilon}$, we have

$$\theta_{v,\,u}(B^{su}(u, e^{-\epsilon})) \subset B^{su}(v, 1) \subset \theta_{v,\,u}(B^{su}(u, e^{\epsilon}))\ .$$

Note that $\widetilde{F}$ is bounded on $\bigcup_{w \in \Gamma K} \pi^{-1}(B(\pi(w), 1))$ if K is compact, since $\widetilde{F}$ is Γ-invariant and continuous. By Equation (142), by Lemma 10.15 and by Lemma 10.13, the result follows from Equation (143). $\square$

Step 5. In this step, in order to work in the direction of giving a lower bound of the left hand side $\int_{T^1M} \frac{I_r\psi}{I_r\rho}\, \rho\, dm^{\nu,\,\mu}$ of Equation (132) by $\int_{T^1M} \psi\, dm_F$ if r is large enough, up to

a positive multiplicative constant, we give a lower bound of $\frac{I_r\psi}{I_r\rho}(u)$ for some r depending on u in a full measure subset for $\Pi = \rho\, dm^{\nu,\mu}$. This lower bound will be improved in the next step.

Recall that we fixed a point x_0 in $\widetilde{M}$. For every $R > 0$, let K_R be the closure of $\pi^{-1}(B(x_0, R)) \cap \widetilde{\Omega}_c\Gamma$, which is a compact subset of $T^1\widetilde{M}$. Let $\widetilde{\Omega}_R$ be the Γ-invariant set of elements $v \in T^1\widetilde{M}$ such that there exists a sequence $(t_i)_{i\in\mathbb{N}}$ in $[0, +\infty[$ converging to $+\infty$ such that $\phi_{-t_i}v \in \Gamma K_R$ for every $i \in \mathbb{N}$. Let Ω_R be the image of $\widetilde{\Omega}_R$ in T^1M. Since the set of elements of $T^1\widetilde{M}$ whose image in T^1M is negatively recurrent under the geodesic flow is the increasing union of the family $(\Omega_R)_{R>0}$, and since ν gives full measure to the negatively recurrent vectors, we fix $R > 0$ large enough so that

$$\Pi(\Omega_R) > 1 - \frac{1}{16N}\ ,$$

where $\Pi = \rho\, dm^{\nu,\mu}$ has been defined at the end of Step 1, and N is given by Lemma 10.10.

Lemma 10.17 *There exist $c_1, c_2 > 0$ such that for every nonnegative $\psi \in \mathscr{C}_c(T^1M;\mathbb{R})$, there exists a Γ-invariant Borel map $r : \widetilde{\Omega}_R \to [0, +\infty[$ such that for every $u' \in \widetilde{\Omega}_R$,*

$$\widetilde{I}_{r(u')}\widetilde{\psi}\,(u') \geq c_1 \Big(\int_{T^1M} \psi\, dm_F \Big)\, \widetilde{I}_{r(u')}\widetilde{\rho}\,(u')$$

and

$$0 < \widetilde{I}_{3\,r(u')}\widetilde{\rho}\,(u') \leq c_2\, \widetilde{I}_{r(u')}\widetilde{\rho}\,(u')\ .$$

Proof. Let $c = \frac{1}{8} \inf_{v\in K_R}\, \widetilde{I}_1 1\,(v)$ and $c' = 8 \sup_{v\in K_R}\, \widetilde{I}_{12} 1\,(v)$, which satisfy $c \leq c'$. Since K_R is a contained in the support of $\widetilde{\Pi} = \widetilde{\rho}\,\widetilde{m}^{\nu,\mu}$, for every $u \in K_R$, we have $\widetilde{I}_{\frac{1}{2}} 1\,(u) > 0$ and $\widetilde{I}_{24} 1\,(u) < +\infty$. By Proposition 10.16 applied with $K = K_R$, $\epsilon = \log 2$, $r = 1$ or $r = 12$, and $\widetilde{\varphi} = 1$, the maps $v \mapsto \widetilde{I}_1 1\,(v)$ and $v \mapsto \widetilde{I}_{12} 1\,(v)$ are hence locally with positive lower bound and locally with finite upper bound on K_R. Since K_R is compact, we have $c > 0$ and $c' < +\infty$. Define

$$c_1 = \frac{c}{c' \int_{T^1M} \rho\, dm_F} \quad \text{and} \quad c_2 = \frac{c'}{c}\ .$$

Let $\varphi = \rho$ (as fixed in the end of Step 1, in particular $\int_{T^1M} \rho\, dm_F > 0$) or $\varphi = \psi$ (as in the statement). By uniform continuity, let us fix $\epsilon \in\,]0, \log 2[$ small enough so that, with the notation introduced before Proposition 10.16, we have

$$\int_{T^1M} \varphi_{\underline{\epsilon}}\, dm_F \geq \frac{1}{2} \int_{T^1M} \varphi\, dm_F \quad \text{and} \quad \int_{T^1M} \rho_{\overline{\epsilon}}\, dm_F \leq 2 \int_{T^1M} \rho\, dm_F\ . \tag{144}$$

Proposition 10.16, applied with this ϵ, with $r = 1$ or $r = 3$, and with $K = K_R$, gives us $r_1, r_2, r_3 > 0$ such that its conclusion holds.

By compactness, let S be a finite subset of K_R such that K_R is covered by the dynamical cells $\mathscr{V}_{r_1,r_2,r_3}(u)$ as u ranges over S.

In the following sequence of inequalities, we use respectively Proposition 10.16 applied to the function $\widetilde{\varphi} \circ \phi_t$, Equation (140) and monotonicity properties, Equation (130) (which is the consequence of the mixing property we are using) with $r = 1$ since S is finite,

and Equation (144): There exists $t_0 \geq 0$ such that for every $t \geq t_0$, for all $u \in \Gamma S$ and $v \in \mathscr{V}_{r_1,r_2,r_3}(u)$,

$$\widetilde{I}_2(\widetilde{\varphi}\circ\phi_t)\,(v) \geq e^{-\epsilon}\;\widetilde{I}_{2e^{-\epsilon}}(\widetilde{\varphi}\circ\phi_t)_{\underline{\epsilon}}\,(u) \geq \frac{1}{2}\;\widetilde{I}_1(\widetilde{\varphi}_{\underline{\epsilon}}\circ\phi_t)\,(u)$$
$$\geq \frac{\widetilde{I}_1 1\,(u)}{4\,\|m_F\|}\int_{T^1M}\varphi_{\underline{\epsilon}}\;dm_F \geq \frac{c}{\|m_F\|}\int_{T^1M}\varphi\;dm_F\;. \tag{145}$$

In particular, $\widetilde{I}_2(\widetilde{\rho}\circ\phi_t)\,(v) > 0$. Similarly, an upper bound is given by

$$\widetilde{I}_6(\widetilde{\rho}\circ\phi_t)\,(v) \leq e^{\epsilon}\;\widetilde{I}_{6e^{\epsilon}}(\widetilde{\rho}\circ\phi_t)_{\overline{\epsilon}}\,(u) \leq 2\;\widetilde{I}_{12}(\widetilde{\rho}_{\overline{\epsilon}}\circ\phi_t)\,(u)$$
$$\leq \frac{4\,\widetilde{I}_{12}1\,(u)}{\|m_F\|}\int_{T^1M}\rho_{\overline{\epsilon}}\;dm_F \leq \frac{c'}{\|m_F\|}\int_{T^1M}\rho\;dm_F\;. \tag{146}$$

Since $\widetilde{I}_2(\widetilde{\rho}\circ\phi_t)\,(v) \leq \widetilde{I}_6(\widetilde{\rho}\circ\phi_t)\,(v) \leq \frac{c'}{\|m_F\|}\int_{T^1M}\rho\;dm_F$, we have, by Equation (145) with $\varphi = \psi$,

$$\widetilde{I}_2(\widetilde{\psi}\circ\phi_t)\,(v) \geq \frac{c}{c'\int_{T^1M}\rho\;dm_F}\int_{T^1M}\psi\;dm_F\;\;\widetilde{I}_2(\widetilde{\rho}\circ\phi_t)\,(v)$$
$$= c_1\int_{T^1M}\psi\;dm_F\;\;\widetilde{I}_2(\widetilde{\rho}\circ\phi_t)(v)\;.$$

By Lemma 10.8 and a cancellation argument, we hence have

$$\widetilde{I}_{2e^t}\widetilde{\psi}\,(\phi_t v) \geq c_1\int_{T^1M}\psi\;dm_F\;\;\widetilde{I}_{2e^t}\widetilde{\rho}\,(\phi_t v)\;.$$

Similarly, since $0 < \widetilde{I}_6(\widetilde{\rho}\circ\phi_t)\,(v) \leq \frac{c'}{c}\;\widetilde{I}_2(\widetilde{\rho}\circ\phi_t)\,(v)$ by Equation (145) and Equation (146), we have

$$0 < \widetilde{I}_{6\,e^t}\widetilde{\rho}\,(\phi_t v) \leq c_2\,\widetilde{I}_{2e^t}\widetilde{\rho}\,(\phi_t v)\;.$$

Since for every $u' \in \widetilde{\Omega}_R$ there exists $t(u') \geq t_0 \geq 0$ (which may be taken to be constant on orbits of Γ and measurable) such that $u' \in \phi_{t(u')}\Gamma K_R$, this concludes the proof of Lemma 10.17, with $r(u') = 2\,e^{t(u')}$. $\Box$

Step 6. In this step, we start to improve the previous one by giving a lower bound of $\frac{I_r\psi}{I_r\rho}(u)$ for u in some good subset $\widetilde{G}_r$ of $T^1\widetilde{M}$, that will be shown in the next step to be large enough for our purpose, if r is large enough.

We first define the good set $\widetilde{G}_r$ on which the lower bound will take place.

For every $s' > 0$, let $\widetilde{\mathscr{E}}_{s'}$ be the Γ-invariant Borel set of $u \in \widetilde{\Omega}_R$ such that $r(u) \leq s'$, where the map r is given by Lemma 10.17 in the previous step. The set $\widetilde{\Omega}_R$ is the increasing union of the family $(\widetilde{\mathscr{E}}_{s'})_{s'>0}$. Note that $\Pi(\Omega_R) > 1-\frac{1}{16N}$ by the choice of R in the previous step. Hence there exist $\sigma > 0$ and a Γ-invariant closed subset $\widetilde{\mathscr{E}}$ of $\widetilde{\mathscr{E}}_\sigma$ whose image $\mathscr{E}$ in T^1M satisfies

$$\Pi(\mathscr{E}) > 1 - \frac{1}{8N}\;.$$

For every $r > 0$, let $\widetilde{\Delta}_r$ be the Γ-invariant Borel set of $u \in \widetilde{\Omega}_R$ such that

$$\widetilde{I}_r(\widetilde{\rho}\,\mathbb{1}_{\widetilde{\mathscr{E}}})\,(u) \leq \frac{1}{2}\;\widetilde{I}_r\widetilde{\rho}\,(u)\;. \tag{147}$$

For every $r > 0$, let $\widetilde{Z}_r$ be the Γ-invariant Borel set of $u \in \widetilde{\Omega}_R$ such that

$$\widetilde{I}_{r+\sigma}\widetilde{\psi}\,(u) > 2\,\widetilde{I}_r\widetilde{\psi}\,(u)\;. \tag{148}$$

Define

$$\widetilde{G}_r = \widetilde{\Omega}_c\Gamma - (\widetilde{\Delta}_r \cup \widetilde{Z}_r)\;,$$

which is a Γ-invariant Borel subset of $T^1\widetilde{M}$. Let Δ_r, Z_r, G_r be the images of $\widetilde{\Delta}_r, \widetilde{Z}_r, \widetilde{G}_r$ in T^1M.

The next result now gives the important property of the set $\widetilde{G}_r$ introduced above, saying that it is indeed a good set concerning the problem of finding a lower bound of $\frac{\widetilde{I}_r\widetilde{\psi}}{\widetilde{I}_r\widetilde{\rho}}(u)$ by a constant times $\int_{T^1M}\psi\,dm_F$.

Lemma 10.18 *There exists $c_3 > 0$ such that for every nonnegative $\psi \in \mathscr{C}_c(T^1M;\mathbb{R})$, for all $r > 0$ and $u \in \widetilde{G}_r$, we have*

$$\widetilde{I}_r\widetilde{\psi}\,(u) \geq c_3 \int_{T^1M}\psi\,dm_F\;\;\widetilde{I}_r\widetilde{\rho}\,(u)$$

Proof. As in [Rob1, page 89], we use a Vitali covering argument summarised in the next statement from loc. cit. (stated here without proof), as pioneered in [Rud] to study the ergodic theory of the strong unstable foliation.

Lemma 10.19 (Vitali) *Let $c > 0$, let X be a compact metric space endowed with a probability measure $\mathbb{P}$ and let $r : X \to\,]0,+\infty[$ be a Borel map such that for every $x \in X$, we have $\mathbb{P}\big(B(x,3\,r(x))\big) \leq c\,\mathbb{P}\big(B(x,r(x))\big)$. Then there exists a finite subset Y of X such that the balls $B_y = B(y,r(y))$ for $y \in Y$ are pairwise disjoint and satisfy*

$$\mathbb{P}\Big(\bigcup_{y\in Y} B_y\Big) \geq \frac{1}{c}\;.$$

Let $r \geq 1$ and $u \in \widetilde{G}_r$. We apply the above Vitali lemma to the compact subset $X = \overline{B^{su}(u,r)} \cap \widetilde{\mathscr{E}}$ endowed with Hamenstädt's distance, to the probability measure $\mathbb{P}$ which is the restriction to X of the measure

$$\widetilde{\rho}(w)\;e^{c_{\widetilde{F}}(w,\,u)}\;d\mu_{W^{su}(u)}(w)$$

(which is not the zero measure since u does not belong to Δ_r) normalised to be a probability measure, to the map $r : X \to [1,+\infty[$ given by Lemma 10.17 in Step 5, and to the constant $c = c_2$ given by the same lemma. The above Vitali lemma, whose hypothesis is satisfied by the second assertion of Lemma 10.17 in Step 5, then gives points $u_1,\dots,u_n$ in X such that the balls $B_i = B(u_i,r(u_i))$ in X satisfy its conclusion. Using respectively

- the fact that u does not belong to $\widetilde{Z}_r$ (see Equation (148)),
- the fact that the balls B_i for $i = 1,\dots,n$ are pairwise disjoint and contained in $B^{su}(u,r+\sigma)$ (since $u_i \in \widetilde{\mathscr{E}}$ implies that $r(u_i) \leq \sigma$ by the definition of $\widetilde{\mathscr{E}}$),
- the equality $W^{su}(u_i) = W^{su}(u)$ and the cocycle property of $c_{\widetilde{F}}$,
- Lemma 10.17 in Step 5 since $\widetilde{\mathscr{E}} \subset \widetilde{\Omega}_R$,
- the cocycle property of $c_{\widetilde{F}}$,

- the conclusion of the above Vitali lemma with $c = c_2$,
- the fact that u does not belong to $\widetilde{\Delta}_r$ (see Equation (147)),

we have

$$\begin{aligned}
\widetilde{I}_r\widetilde{\psi}\,(u) &\geq \frac{1}{2}\,\widetilde{I}_{r+\sigma}\widetilde{\psi}\,(u) \geq \frac{1}{2}\,\sum_{i=1}^{n}\int_{w\in B_i}\widetilde{\psi}(w)\;e^{c_{\widetilde{F}}(w,\,u)}\;d\mu_{W^{su}(u)}(w)\\
&= \frac{1}{2}\,\sum_{i=1}^{n}\;\widetilde{I}_{r(u_i)}\widetilde{\psi}\,(u_i)\;e^{c_{\widetilde{F}}(u_i,u)}\\
&\geq \frac{c_1}{2}\;\int_{T^1M}\psi\;dm_F\;\sum_{i=1}^{n}\;\widetilde{I}_{r(u_i)}\widetilde{\rho}\,(u_i)\;e^{c_{\widetilde{F}}(u_i,u)}\\
&= \frac{c_1}{2}\;\int_{T^1M}\psi\;dm_F\;\sum_{i=1}^{n}\int_{w\in B_i}\widetilde{\rho}(w)\;e^{c_{\widetilde{F}}(w,\,u)}\;d\mu_{W^{su}(u)}(w)\\
&\geq \frac{c_1}{2\,c_2}\;\int_{T^1M}\psi\;dm_F\;\int_{w\in\overline{B^{su}(u,\,r)}\,\cap\,\widetilde{\mathscr{E}}}\widetilde{\rho}(w)\;e^{c_{\widetilde{F}}(w,\,u)}\;d\mu_{W^{su}(u)}(w)\\
&\geq \frac{c_1}{4\,c_2}\;\int_{T^1M}\psi\;dm_F\;\;\widetilde{I}_r\widetilde{\rho}\,(u)\;.
\end{aligned}$$

This proves Lemma 10.18 with $c_3 = \frac{c_1}{4\,c_2}$. $\square$

Step 7. In this penultimate step, we conclude the search of a lower bound of $\frac{\widetilde{I}_r\widetilde{\psi}}{\widetilde{I}_r\widetilde{\rho}}(u)$ for r large enough, on a large enough subset of $u \in T^1\widetilde{M}$, in order to obtain a lower bound of the left hand side $\int_{T^1M}\frac{I_r\psi}{I_r\rho}\;\rho\;dm^{\nu,\,\mu}$ of Equation (132) by a positive constant times $\int_{T^1M}\psi\;dm_F$.

We start by proving that the good set G_r has large measure for some r large enough, or equivalently that the bad sets Z_r and Δ_r have small measures.

Lemma 10.20 *For every $r \geq 1$, we have $\Pi(\Delta_r) \leq \frac{1}{4}$.*

Proof. By the crucial adjointness property of the operators I_r and J_r (see Lemma 10.9 in Step 2), we have

$$\int_{T^1M} I_r(\rho\,\mathbb{1}_{\,{}^c\mathscr{E}})\;\frac{\rho}{I_r\rho}\;\mathbb{1}_{\,\Delta_r}\;dm^{\nu,\,\mu} = \int_{T^1M}\rho\,\mathbb{1}_{\,{}^c\mathscr{E}}\;J_r\Big(\frac{\rho}{I_r\rho}\;\mathbb{1}_{\,\Delta_r}\Big)\;dm^{\nu,\,\mu}\;. \tag{149}$$

Since $\mathbb{1}_{\,{}^c\mathscr{E}} = 1 - \mathbb{1}_{\mathscr{E}}$, the definition of Δ_r (see Equation (147)) implies that for every $u \in \Delta_r$, we have

$$\frac{I_r(\rho\,\mathbb{1}_{\,{}^c\mathscr{E}})}{I_r\rho}(u) = 1 - \frac{I_r(\rho\,\mathbb{1}_{\mathscr{E}})}{I_r\rho}(u) \geq \frac{1}{2}\;.$$

Hence the left hand side of Equation (149) is bounded from below by $\frac{1}{2}\;\int_{T^1M}\rho\,\mathbb{1}_{\,\Delta_r}\;dm^{\nu,\,\mu}$ $= \frac{1}{2}\;\Pi(\Delta_r)$. Since $\mathbb{1}_{\,\Delta_r} \leq 1$, since J_r is a positive operator, and by Lemma 10.11 in Step 3, the right hand side of Equation (149) is bounded from above by $N\,\Pi({}^c\mathscr{E})$, which is at most $\frac{1}{8}$ by the definition of $\mathscr{E}$ in the beginning of Step 6. This proves the result. $\square$

Before proving that the bad set Z_r also has small measure for some r large enough, we start by showing that the map $r \mapsto \widetilde{I}_r\widetilde{\varphi}\,(u)$, defined by integrating nonnegative Borel maps

$\widetilde{\varphi}$ (for the strong unstable measure weighted by the cocycle $c_{\widetilde{F}}$) on strong unstable balls, has subexponential growth in the radius of the ball. When $\widetilde{F}$ is bounded, the proof below even proves its polynomial growth, recovering with a different presentation the result of Roblin when $\widetilde{F} = 0$.

Lemma 10.21 *For every $u \in T^1M$ and for every bounded nonnegative measurable map $\varphi : T^1M \to \mathbb{R}$, there exists $c_u > 0$ such that, as r tends to $+\infty$,*

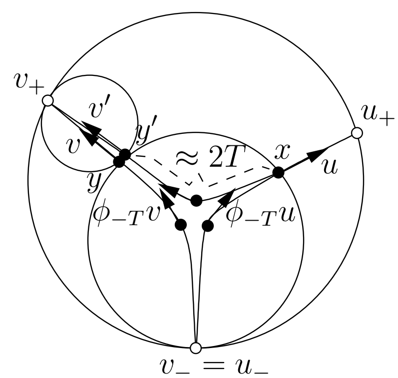
$$I_r\varphi\,(u) = \mathrm{O}(e^{c_u\,(\log r)^2})\ .$$

Proof. We may assume that φ is constant with value 1. We fix $u \in T^1\widetilde{M}$ and we allow v to vary in $W^{su}(u)$. Define $x = \pi(u)$ and $y = \pi(v)$. By the definition of $\mu_{W^{su}(u)}$ (see Equation (42)), we have

$$e^{c_{\widetilde{F}}(v,\,u)}\,d\mu_{W^{su}(u)}(v) = e^{c_{\widetilde{F}}(v,\,u)}e^{C_{F-\delta,\,v_+}(x,\,y)}\,d\mu_x(v_+)\ .$$

Since the Patterson measure μ_x is finite, we hence only have to prove that, as v goes to infinity in $W^{su}(u)$, we have

$$c_{\widetilde{F}}(v,u) + C_{F-\delta,\,v_+}(x,y) = \mathrm{O}\left((\log d_{W^{su}(u)}(u,v))^2\right)\ .$$



By the convexity of horoballs, there exists $v' \in W^{ss}(v)$ which is tangent to the geodesic ray from x to v_+. Let $y' = \pi(v')$. By the properties of geodesic triangles in CAT(-1) spaces (or the techniques of approximation by trees), there exists a universal constant $c'' \geq 0$ such that if $T = \log d_{W^{su}(u)}(u,v)$ is large enough, then

(i) $d_{W^{ss}(\phi_{-T}v)}(\phi_{-T}v, \phi_{-T}v') \leq c''$,

(ii) $d_{W^{su}(\phi_{-T}v)}(\phi_{-T}v, \phi_{-T}u) \leq c''$,

(iii) $|\,d(y',x) - 2T\,| \leq c''$.

By the definition of the cocycles, we have

$$\begin{aligned}
c_{\widetilde{F}}(v,u) + C_{F-\delta,\,v_+}(x,y) &= \int_0^{+\infty} \widetilde{F}(\phi_{-t}v) - \widetilde{F}(\phi_{-t}u)\,dt + \int_0^{+\infty} \widetilde{F}(\phi_t v) - \widetilde{F}(\phi_t v')\,dt \\
&\quad - \int_{-d(y',\,x)}^{0} \big(\widetilde{F}(\phi_t v') - \delta\big)\,dt \\
&= \int_T^{+\infty} \widetilde{F}(\phi_{-t}v) - \widetilde{F}(\phi_{-t}u)\,dt + \int_{-T}^{+\infty} \widetilde{F}(\phi_t v) - \widetilde{F}(\phi_t v')\,dt \\
&\quad + \delta\,d(y',x) - \int_{-d(y',\,x)}^{-T} \widetilde{F}(\phi_t v')\,dt - \int_0^{T} \widetilde{F}(\phi_{-t}u)\,dt\ .
\end{aligned}$$

Let $c \geq 0$ and $\alpha \in\,]0,1]$ be the Hölder constants of $\widetilde{F}$, as in the beginning of the proof of Lemma 10.13. By Equation (9) and Equation (8), there exists $c' > 0$ such that

$$d(\phi_{-t}v, \phi_{-t}u) \leq c' d_{W^{su}(\phi_{-t}v)}(\phi_{-t}v, \phi_{-t}u) = c' e^{T-t} d_{W^{su}(\phi_{-T}v)}(\phi_{-T}v, \phi_{-T}u)\ .$$

By (ii), we hence have

$$\int_T^{+\infty} \widetilde{F}(\phi_{-t}v) - \widetilde{F}(\phi_{-t}u)\;dt \leq \frac{c}{\alpha}(c'c'')^\alpha\;.$$

Similarly by (i), we have

$$\int_{-T}^{+\infty} \widetilde{F}(\phi_t v) - \widetilde{F}(\phi_t v')\;dt \leq \frac{c}{\alpha}(c'c'')^\alpha\;.$$

As mentioned in the remark at the end of Subsection 2.1, by the Hölder-continuity of $\widetilde{F}$, we have

$$|\,\widetilde{F}(w) - \widetilde{F}(w')\,| \leq 3\,c\,d(w,w') + c$$

for all $w, w' \in T^1\widetilde{M}$. Hence by (iii) and since $d(\phi_t w, w) = |t|$ for every $w \in T^1\widetilde{M}$, we have

$$\begin{aligned}
&|\,c_{\widetilde{F}}(v,u) + C_{F-\delta,\,v_+}(x,y)\,| \\
\leq\;& 2\frac{c}{\alpha}(c'c'')^\alpha + \delta\,(2T + c'') + 2\,(T + c'')\Big(\max_{\pi^{-1}(\{x\})} |\widetilde{F}| + 3c\,(T + c'') + c\Big)\;.
\end{aligned}$$

The result follows. □

Let us now prove that the bad set Z_r also has small measure for some r large enough.

Lemma 10.22 *For every $r_0 \geq 1$, there exists $r \geq r_0$ such that $\Pi(Z_r) \leq \frac{1}{4}$.*

Proof. If the interior of the support of ψ does not meet $\Omega_c\Gamma$, then Z_r is empty, and the result is true. We hence assume that the interior of the support of ψ meets $\Omega_c\Gamma$.

Let us fix $u \in \widetilde{\Omega}_c\Gamma$. For every $t \geq 0$, let $n(t)$ be the number of integers k in $[0, \lfloor \frac{t}{\sigma} \rfloor]$ such that there exists $r \in [k\sigma, (k+1)\sigma[$ with $u \in \widetilde{Z}_r$. Hence the period of time spent by u in the sets $\widetilde{Z}_r$ as r ranges from 0 to t, that is $\int_0^t \mathbb{1}_{\widetilde{Z}_r}(u)\;dr$, is at most $n(t)\,\sigma$. To prove Lemma 10.22, we will first prove that $n(t)$ grows slowly in t.

Recall without proof the following result of [Dal2] (generalising, to our (almost no) assumptions on $\widetilde{M}$ and Γ, results of Hedlund and others), which is valid since the geodesic flow is mixing, hence topologically mixing on $\Omega\Gamma$ by Babillot's Theorem 8.1.

Proposition 10.23 (Dal'Bo) *For every $v \in \widetilde{\Omega}_c\Gamma$, the intersection with $\widetilde{\Omega}_c\Gamma$ of the orbit $\Gamma W^{su}(v)$ under Γ of the strong unstable leaf of v is dense in $\widetilde{\Omega}_c\Gamma$.*

Hence $W^{su}(u)$ meets the (nonempty) intersection with $\widetilde{\Omega}_c\Gamma$ of the interior of the support of $\widetilde{\psi}$. In particular, there exists $t_0 \geq 1$ such that $\widetilde{I}_{t_0}\widetilde{\psi}\,(u) > 0$. As the hinted proof in [Rob1, page 91] is slightly incorrect, let us prove the following claim.

Claim. There exists $c > 0$ (depending on u) such that for every $t \geq t_0$,

$$\widetilde{I}_{t+2\sigma}\widetilde{\psi}\,(u) \geq c\,2^{\frac{n(t)}{2}}\;.$$

Proof. To simplify the notation, let f be the nondecreasing map $r \mapsto \widetilde{I}_r\widetilde{\psi}\,(u)$, and for every $t \geq t_0$, let $n = n(t)$. Let $0 \leq k_1 < k_2 < \cdots < k_n \leq \lfloor \frac{t}{\sigma} \rfloor$ and $r_i \in [k_i\sigma, (k_i+1)\sigma[$ such

that $u \in \widetilde{Z}_{r_i}$. Note that $r_{i+2} \geq r_i + \sigma$ (but $r_{i+1} - r_i$ could be strictly less than σ). Let $p = \lfloor \frac{t_0}{\sigma} \rfloor + 2$ and $q = \lfloor \frac{n-p}{2} \rfloor$. Note that $n - 2q - 1 \geq p - 1 \geq 0$ and

$$r_{n-2q} \geq k_{n-2q}\, \sigma \geq (n-2q-1)\, \sigma \geq (p-1)\, \sigma \geq t_0 \ .$$

By applying $q+1$ times the definition of the sets $\widetilde{Z}_r$ (see Equation (148)) and by monotonicity, we have

$$\begin{aligned} f(t+2\sigma) &\geq f(r_n+\sigma) \geq 2\, f(r_n) \geq f(r_{n-2}+\sigma) \geq \cdots \geq 2^q f(r_{n-2q}+\sigma) \\ &\geq 2^{q+1} f(r_{n-2q}) \geq 2^{\frac{n-p}{2}}\, f(t_0) \ . \end{aligned}$$

Hence the result follows with $c = 2^{-\frac{p}{2}}\, f(t_0) > 0$. $\Box$

This claim and Lemma 10.21 imply the slow growth property of n that $n(t) = \mathrm{O}((\log t)^2)$ as $t \to +\infty$. Therefore, for every $u \in \widetilde{\Omega}_c\Gamma$, we have, by the definition of the map $t \mapsto n(t)$,

$$\lim_{t \to +\infty} \frac{1}{t} \int_0^t \mathbb{1}_{\widetilde{Z}_r}(u)\; dr = 0 \ .$$

Since $\Omega_c\Gamma$ has full measure with respect to the probability measure Π, and by Lebesgue's dominated convergence theorem, we have

$$\lim_{t \to +\infty} \frac{1}{t} \int_0^t \Pi(Z_r)\; dr = 0 \ .$$

Lemma 10.22 now follows immediately. $\Box$

Let us now conclude the minoration of the left hand side $\int_{T^1M} \frac{I_r\psi}{I_r\rho}\, \rho\; dm^{\nu,\mu}$ of Equation (132) by a positive constant times $\int_{T^1M} \psi\; dm_F$.

By Lemma 10.22 and Lemma 10.20, since $G_r = \Omega_c\Gamma - (\Delta_r \cup Z_r)$ and since $\Omega_c\Gamma$ has full measure with respect to $\Pi = \rho\, m^{\nu,\mu}$ by Lemma 10.7, there exists $r \geq 1$ large enough such that

$$\Pi(G_r) \geq 1 - \Pi(Z_r) - \Pi(\Delta_r) \geq \frac{1}{2} \ .$$

Hence, using Lemma 10.18, we have

$$\begin{aligned} \int_{T^1M} \frac{I_r\psi}{I_r\rho}\, \rho\; dm^{\nu,\mu} &= \int_{\Omega_c\Gamma} \frac{I_r\psi}{I_r\rho}\; d\Pi \geq \int_{G_r} \frac{I_r\psi}{I_r\rho}\; d\Pi \\ &\geq c_3\, \Pi(G_r) \int_{T^1M} \psi\; dm_F \geq \frac{c_3}{2} \int_{T^1M} \psi\; dm_F \ , \end{aligned} \tag{150}$$

which is the lower bound we were looking for.

Step 8. This step, which is an easy ergodicity argument, is the final one to conclude the proof of Theorem 10.4.

By Equation (132), Equation (150) and Equation (133), with $c_4 = \frac{2\,N}{c_3}$, we have, for every nonnegative $\psi \in \mathscr{C}_c(T^1M;\mathbb{R})$,

$$\int_{T^1M} \psi\; dm_F \leq c_4 \int_{T^1M} \psi\; dm^{\nu,\mu} \ .$$

Hence $m_F \leq c_4\, m^{\nu,\mu}$, and $m_F = m^{\mu^\iota,\mu}$ is absolutely continuous with respect to $m^{\nu,\mu}$. By disintegration (see Lemma 10.7 in Step 1), for every transversal T to the strong unstable foliation, the measure μ^ι_T is hence absolutely continuous with respect to ν_T. Note that this is true for any Γ-equivariant $c_{\widetilde{F}}$-quasi-invariant transverse measure ν which gives full support to the negatively recurrent set.

This implies in particular that the Γ-equivariant $c_{\widetilde{F}}$-quasi-invariant transverse measure $\mu^\iota = (\mu^\iota_T)_{T\in\mathscr{T}(\widetilde{\mathscr{W}}^{su})}$ is ergodic. Indeed, let A be a Γ-invariant $\widetilde{\mathscr{W}}^{su}$-saturated (that is, which is a union of strong unstable leaves) subset of $T^1\widetilde{M}$, whose intersection with any transversal is measurable, and assume that $\mu^\iota_T(A\cap T)>0$ for at least one transversal T. Then $(\mathbb{1}_{A\cap T}\,\mu^\iota_T)_{T\in\mathscr{T}(\widetilde{\mathscr{W}}^{su})}$ is also a Γ-equivariant $c_{\widetilde{F}}$-quasi-invariant transverse measure for the strong unstable foliation. By the above absolute continuity claim applied to this family, the measure μ^ι_T is absolutely continuous with respect to $\mathbb{1}_{A\cap T}\,\mu^\iota_T$, hence $\mu^\iota_T({}^cA\cap T)=0$ for all transversals T.

Now, by ergodicity, for every transversal T to the strong unstable foliation, there exists a constant $c_T\geq 0$ such that $\mu^\iota_T = c_T\nu_T$. Since μ^ι and ν are quasi-invariant under holonomy with respect to the same cocycle $c_{\widetilde{F}}$, by the density property of strong unstable leaves recalled in Proposition 10.23, and since $T\cap\widetilde{\Omega}_c\Gamma$ has full measure with respect to both μ^ι_T and ν_T for all transversals T, the constant c_T is independant of T. Hence μ^ι and ν are proportional, which concludes the proof of Theorem 10.4. □

To conclude Chapter 10, we obtain a complete classification of the Γ-equivariant $c_{\widetilde{F}}$-quasi-invariant transverse measures for the strong unstable foliation of $T^1\widetilde{M}$, under the assumption that Γ is geometrically finite. We refer to the beginning of Subsection 8.2 for a definition of a geometrically finite group of isometries of $\widetilde{M}$.

The additional measures are the following ones. Let $\widetilde{W}$ be a strong unstable leaf in $T^1\widetilde{M}$ such that the family $(\gamma\widetilde{W})_{\gamma\in\Gamma}$ is locally finite (that is, for every compact subset K of $T^1\widetilde{M}$, the set of $\gamma\in\Gamma$ such that $\gamma\widetilde{W}$ meets K is finite), or, equivalently, such that the image of $\widetilde{W}$ in T^1M is a closed subset of T^1M. Fix $v\in\widetilde{W}$ and let $\Gamma_{\widetilde{W}}$ be the stabiliser of $\widetilde{W}$ in Γ. We denote by $\mathscr{D}_z$ the unit Dirac mass at a point $z\in T^1\widetilde{M}$. For every transversal T to $\widetilde{\mathscr{W}}^{su}$, the measure

$$\nu^{\widetilde{W}}_T = \sum_{\gamma\in\Gamma/\Gamma_{\widetilde{W}},\; w\in T\cap\gamma\widetilde{W}} e^{c_{\widetilde{F}}(w,\,v)}\;\mathscr{D}_w$$

is locally finite, and $(\nu^{\widetilde{W}}_T)_{T\in\mathscr{T}(\widetilde{\mathscr{W}}^{su})}$ is a nonzero Γ-equivariant family of locally finite measures on transversals to the strong unstable foliation, which is stable by restrictions, and satisfies the property (iii) of $c_{\widetilde{F}}$-quasi-invariant transverse measures. Note that replacing v by another element of $\widetilde{W}$ changes this family only by a multiplicative constant. Note that, since $\widetilde{W}=W^{su}(v)$, the (discrete, countable) support of $\nu^{\widetilde{W}}_T$ is the set $\{w\in T\;:\;w_-\in\Gamma v_-\}$. When Γ is torsion free, if T is a transversal to the strong unstable foliation $\mathscr{W}^{su}$ on T^1M and if W is a leaf of $\mathscr{W}^{su}$ which is a closed subset of T^1M, the associated measure on T is

$$\sum_{w\in T\cap W} e^{c_F(w,\,v)}\;\mathscr{D}_w$$

where $c_F(w,v)=\int_0^{+\infty}F(\phi_{-t}w)-F(\phi_{-t}v)\,dt$ (see Equation (80)).

Corollary 10.24 *Under the assumptions of Theorem 10.4, assume furthermore that Γ is geometrically finite. Then, up to a multiplicative constant, the only ergodic Γ-equivariant $c_{\widetilde{F}}$-quasi-invariant transverse measures for the strong unstable foliation on $T^1\widetilde{M}$ are*

- *the family $(\mu^{\iota}_T)_{T\in\mathscr{T}(\widetilde{\mathscr{W}}^{su})}$ or*
- *the families $(\nu^{\widetilde{W}}_T)_{T\in\mathscr{T}(\widetilde{\mathscr{W}}^{su})}$ where $\widetilde{W}$ is a strong unstable leaf of $T^1\widetilde{M}$ whose image in T^1M is a closed subset.*

Proof. Since Γ is geometrically finite, a point ξ in $\partial_\infty\widetilde{M} - \Lambda_c\Gamma$ is either a point in $\partial_\infty\widetilde{M} - \Lambda\Gamma$ (in which case its stabiliser in Γ is finite) or a (bounded) parabolic limit point (in which case its stabiliser in Γ is infinite). In both cases, the image in M of any horosphere centered at ξ is a closed subset: This follows from the fact that there exists, in both cases, a horoball centered at ξ which is precisely invariant under Γ (see the beginning of Subsection 8.2 for the second case). In particular, for every $w \in T^1\widetilde{M}$ such that $w_- \notin \Lambda_c\Gamma$, the image in T^1M of the strong unstable leaf $W^{su}(w)$ is a closed subset (see also [Dal2] for further information on the topology of the strong unstable manifolds in M).

Let $\nu = (\nu_T)_{T\in\mathscr{T}(\widetilde{\mathscr{W}}^{su})}$ be an ergodic Γ-equivariant $c_{\widetilde{F}}$-quasi-invariant transverse measure for $\widetilde{\mathscr{W}}^{su}$.

Assume first that there exists a transversal T to $\widetilde{\mathscr{W}}^{su}$ such that the measurable set $A = \{w \in T \;:\; w_- \notin \Lambda_c\Gamma\}$ has positive measure with respect to ν_T. Then by ergodicity, the set A, which is saturated by (the intersections with T of) the leaves of $\widetilde{\mathscr{W}}^{su}$, has full measure. Furthermore, there exists $w \in A$ such that the support of ν_T is the closure in T of the intersection with T of $\Gamma W^{su}(w)$. Since $w_- \in \Lambda_c\Gamma$, this intersection is already closed, hence by quasi-invariance under holonomy of ν, with $\widetilde{W} = W^{su}(w)$, we have that $(\nu_T)_{T\in\mathscr{T}(\widetilde{\mathscr{W}}^{su})}$ is a multiple of $(\nu^{\widetilde{W}}_T)_{T\in\mathscr{T}(\widetilde{\mathscr{W}}^{su})}$.

Assume now on the contrary that ν gives full measure to the negatively recurrent set. Then the result follows from Theorem 10.4. □

11 Gibbs states on amenable covers

Let $(\widetilde{M}, \Gamma, \widetilde{F})$ be as in the beginning of Chapter 2: $\widetilde{M}$ is a complete simply connected Riemannian manifold, with dimension at least 2 and pinched sectional curvature at most -1, Γ is a nonelementary discrete group of isometries of $\widetilde{M}$, and $\widetilde{F} : T^1\widetilde{M} \to \mathbb{R}$ is a Hölder-continuous Γ-invariant map. Fix $x_0 \in \widetilde{M}$.

A subgroup Γ_0 of the isometry group $\mathrm{Isom}(\widetilde{M})$ of $\widetilde{M}$ is said *to normalise* Γ if it is contained in the *normaliser*

$$N(\Gamma) = \{\gamma \in \mathrm{Isom}(\widetilde{M}) \;:\; \gamma\Gamma\gamma^{-1} = \Gamma\}$$

of Γ in $\mathrm{Isom}(\widetilde{M})$.

We study in this chapter the behaviour of the critical exponents, the Patterson densities and the Gibbs measures associated to normal subgroups of Γ. The main point is that this gives a natural framework in which one can study precisely these objects when the Gibbs measure is not finite. We prove results analogous to those in the chapters 3, 4, 5, 8, 10, underlining the fact that, contrarily to most of these chapters, the Gibbs measures may be infinite. Most of the results are extensions of those when $F = 0$ in [Rob3], with

similar proofs, and we will concentrate on the new features. This paper of Roblin furthermore mentions in its introduction that its contents remain valid for Patterson densities with potentials (referring to the multiplicative approach of Ledrappier-Coudene, see the beginning of Chapter 3 for an explanation of the differences with the additive convention considered in this text).

11.1 Improving Mohsen's shadow lemma

Mohsen's shadow lemma 3.10 for a Patterson density of (Γ, F), as its original version by Sullivan [Sul1] when $F = 0$ (see [Rob1, p. 10]), requires the balls whose shadows are considered to be centered at special points. The following result, due to [Rob3, Théo. 1.1.1] when $F = 0$, improves the range of validity of Mohsen's shadow lemma, in a uniform way. It will be useful to study Patterson densities on Galois covers of M. Its proof follows those of the above works of Sullivan, Mohsen and Roblin.

Proposition 11.1 *Let Γ_0 be a subgroup of $\operatorname{Isom}(\widetilde{M})$, which contains and normalises Γ, such that $\widetilde{F}$ is Γ_0-invariant. For every compact subset K of $\widetilde{M}$, there exist nondecreasing maps $R = R_K$ and $C = C_K$ from $[\delta_{\Gamma,F}, +\infty[$ to $]0, +\infty[$ such that for every $\sigma \geq \delta_{\Gamma,F}$, for every Patterson density $(\mu_x)_{x\in\widetilde{M}}$ for (Γ, F) of dimension σ, for all $x, y \in \Gamma_0 K$, we have*

$$\frac{1}{C(\sigma)}\,\|\mu_y\|\,e^{\int_x^y(\widetilde{F}-\sigma)} \leq \mu_x\big(\mathscr{O}_x B(y, R(\sigma))\big) \leq C(\sigma)\,\|\mu_y\|\,e^{\int_x^y(\widetilde{F}-\sigma)}\,. \tag{151}$$

Proof. We start by the following key observation, which will be used several times in this chapter.

Lemma 11.2 *For every $\alpha \in N(\Gamma)$, if $(\mu_x)_{x\in\widetilde{M}}$ is a Patterson density for (Γ, F) of dimension σ, then so is $(\alpha_*^{-1}\mu_{\alpha x})_{x\in\widetilde{M}}$.*

Proof. For all $\gamma \in \Gamma$ and $x \in \widetilde{M}$, since $\alpha\gamma\alpha^{-1} \in \Gamma$, we have

$$\gamma_*\alpha_*^{-1}\mu_{\alpha x} = \alpha_*^{-1}(\alpha\gamma\alpha^{-1})_*\mu_{\alpha x} = \alpha_*^{-1}\mu_{\alpha\gamma x}\,,$$

and for all $x, y \in \widetilde{M}$ and $\xi \in \partial_\infty\widetilde{M}$, we have

$$\frac{d\,\alpha_*^{-1}\mu_{\alpha x}}{d\,\alpha_*^{-1}\mu_{\alpha y}}(\xi) = \frac{d\mu_{\alpha x}}{d\mu_{\alpha y}}(\alpha\xi) = e^{-C_{F-\sigma,\,\alpha\xi}(\alpha x,\alpha y)} = e^{-C_{F-\sigma,\,\xi}(x,y)}\,,$$

since $\widetilde{F}$ is Γ_0-invariant. □

Now, let K be a fixed compact subset of $\widetilde{M}$. Since the statement is empty if $\delta_{\Gamma,F} = +\infty$, we assume that $\delta_{\Gamma,F} < +\infty$. By the above observation, we only have to prove Equation (151) when $x \in K$, since the validity of Equation (151) when $x \in \alpha K$ for any $\alpha \in \Gamma_0$ is obtained by replacing $(\mu_x)_{x\in\widetilde{M}}$ by $(\alpha_*^{-1}\mu_{\alpha x})_{x\in\widetilde{M}}$ and y by $\alpha^{-1}y$.

Let us prove that there exist non-decreasing maps $R', C' : [\delta_{\Gamma,F}, +\infty[\,\to\,]0, +\infty[$ such that for every $\sigma \geq \delta_{\Gamma,F}$, for every Patterson density $(\mu_x)_{x\in\widetilde{M}}$ for (Γ, F) of dimension σ, for all $x \in K$ and $y \in \Gamma_0 K$, we have

$$\frac{1}{C'(\sigma)}\,\|\mu_y\| \leq \mu_y(\mathscr{O}_x B(y, R'(\sigma))) \leq C'(\sigma)\,\|\mu_y\|\,. \tag{152}$$

To prove that Equation (152) implies Equation (151), note that we have, by Equation (35),

$$\mu_x(\mathscr{O}_x B(y, R'(\sigma))) = \int_{\xi\in\mathscr{O}_x B(y,\, R'(\sigma))} e^{-C_{F-\sigma,\,\xi}(x,\,y)}\; d\mu_y(\xi)\;.$$

By Lemma 3.4 (2) applied with $r = r_0 = R'(\sigma)$, since $\widetilde{F}$ is Γ_0-invariant, hence is uniformly bounded on $\pi^{-1}(B(y, R'(\sigma)))$ for $y \in \Gamma_0 K$, there exists $C''(\sigma) > 0$ (depending only on r, and that may be taken to be nondecreasing in σ) such that for all $x \in K$ and $\xi \in \mathscr{O}_x B(y, R'(\sigma))$,

$$\Big|\, C_{F-\sigma,\,\xi}(x, y) + \int_x^y (\widetilde{F} - \sigma)\, \Big| \leq C''(\sigma)\;.$$

Hence

$$\begin{aligned} e^{-C''(\sigma)}\; e^{\int_x^y (\widetilde{F}-\sigma)}\; \mu_y(\mathscr{O}_x B(y, R'(\sigma))) &\leq \mu_x(\mathscr{O}_x B(y, R'(\sigma))) \\ &\leq e^{C''(\sigma)}\; e^{\int_x^y (\widetilde{F}-\sigma)}\; \mu_y(\mathscr{O}_x B(y, R'(\sigma)))\;. \end{aligned}$$

Therefore the result follows with $R = R'$ and $C = C' e^{C''}$.

Let us now prove the claim involving Equation (152).

The upper bound in this equation is clear with $C'(\sigma) = 1$, whatever $R'(\sigma)$ is. For future use, this proves that for every $R > 0$, for every compact subset K of $\widetilde{M}$, there exists $c > 0$ (depending only on K and R) such that, for all $x, y \in \Gamma_0 K$, we have

$$\mu_x(\mathscr{O}_x B(y, R)) \leq c\, \|\mu_y\|\; e^{\int_x^y (\widetilde{F}-\sigma)}\;. \tag{153}$$

To prove the lower bound in Equation (152), we assume for a contradiction that there exist sequences $(x_i)_{i\in\mathbb{N}}$ and $(x'_i)_{i\in\mathbb{N}}$ in K, $(R_i)_{i\in\mathbb{N}}$ in $]0, +\infty[$ converging to $+\infty$, $(\alpha_i)_{i\in\mathbb{N}}$ in Γ_0 and $(\mu^i)_{i\in\mathbb{N}}$, where μ^i is a Patterson density for (Γ, F) with bounded dimension σ_i such that

$$\lim_{i\to+\infty} \frac{1}{\|\mu^i_{\alpha_i x_i}\|}\; \mu^i_{\alpha_i x_i}(\mathscr{O}_{x'_i} B(\alpha_i x_i, R_i)) \;= 0\;. \tag{154}$$

Up to extraction, we have $\lim x_i = x \in K$, $\lim x'_i = x' \in K$, $\lim \sigma_i = \sigma \geq \delta_{\Gamma,\,F}$ and $\lim \alpha_i^{-1} x'_i = \xi \in \widetilde{M} \cup \partial_\infty \widetilde{M}$. By Equation (154), the shadow $\mathscr{O}_{x'_i} B(\alpha_i x_i, R_i)$ is different from $\partial_\infty \widetilde{M}$, hence $d(x'_i, \alpha_i x_i) \geq R_i$, which tends to $+\infty$ as $i \to +\infty$. Therefore $\xi \in \partial_\infty \widetilde{M}$. Let

$$\nu^i = \frac{1}{\|\mu^i_{\alpha_i x_i}\|}\; (\alpha_i^{-1})_* \mu^i_{\alpha_i x_i}\;,$$

which is a probability measure on the compact metrisable space $\partial_\infty \widetilde{M}$. By Banach-Alaoglu's theorem, up to extraction, the sequence $(\nu^i)_{i\in\mathbb{N}}$ weak star converges to a probability measure ν_x. Define, for every $z \in \widetilde{M}$, a measure ν_z on $\partial_\infty \widetilde{M}$ by

$$d\nu_z(\xi) = e^{-C_{F-\sigma,\,\xi}(z,\,x)}\; d\nu_x(\xi)\;.$$

By the continuity of the Gibbs cocycle and by taking limits, the family $(\nu_z)_{z\in\widetilde{M}}$ is a Patterson density for (Γ, F) of dimension σ.

Let V be an relatively compact open subset of $\partial_\infty \widetilde{M} - \{\xi\}$. Then V is contained in $\mathscr{O}_{\alpha_i^{-1} x'_i} B(x_i, R_i)$ for i large enough, since $\lim R_i = +\infty$ and $(x_i)_{i\in\mathbb{N}}$ stays in the compact subset K. Hence

$$\nu_x(V) \leq \limsup_{i\to+\infty} \nu^i(\mathscr{O}_{\alpha_i^{-1} x'_i} B(x_i, R_i)) = 0$$

by Equation (154). Therefore the measure ν_x is supported in $\{\xi\}$. Since the support of a Patterson density of (Γ, F) is Γ-invariant, this implies that ξ is fixed by Γ. This is a contradiction since Γ is nonelementary. $\Box$

The following consequence is due to [Rob3, Lem. 1.2.4] when $F = 0$.

Corollary 11.3 *Let Γ_0 be a discrete subgroup of $\mathrm{Isom}(\widetilde{M})$, which contains and normalises Γ, such that $\widetilde{F}$ is Γ_0-invariant. Let $\sigma \in \mathbb{R}$ and let $(\mu_x)_{x\in\widetilde{M}}$ be a Patterson density for (Γ, F) of dimension σ.*

Then the critical exponent of the series $Q'(s) = \sum_{y\in\Gamma_0 x_0} \|\mu_y\|\; e^{\int_{x_0}^{y}(\widetilde{F}-s)}$ is at most σ. If μ_{x_0} gives positive measure to $\Lambda_c\Gamma_0$, then this critical exponent is equal to σ, and this series diverges at $s = \sigma$.

Proof. Note that $\sigma \geq \delta_{\Gamma,F}$ by Corollary 3.11 (2). Let $R = R_{\{x_0\}}(\sigma)$ and $C = C_{\{x_0\}}(\sigma)$ be given by Proposition 11.1 for $K = \{x_0\}$. For every $n \in \mathbb{N}$, let

$$E_n = \{y \in \Gamma_0 x_0 \;:\; n \leq d(x_0, y) < n+1\}\;.$$

As seen in the proof of Corollary 3.11, since Γ_0 is discrete, there exists $\kappa \in \mathbb{N}$ such that for every $n \in \mathbb{N}$, any element of $\partial_\infty\widetilde{M}$ belongs to at most κ elements of the family $\big(\mathscr{O}_{x_0}B(y,R)\big)_{y\in E_n}$.

By the lower bound in Equation (151) in Proposition 11.1, for all $n \in \mathbb{N}$, we have

$$\|\mu_{x_0}\| \geq \frac{1}{\kappa}\sum_{y\in E_n}\mu_{x_0}(\mathscr{O}_{x_0}B(y,R)) \geq \frac{1}{C\kappa}\sum_{y\in E_n}\|\mu_y\|\; e^{\int_{x_0}^{y}(\widetilde{F}-\sigma)}\;.$$

Hence if $s > \sigma$, then the series $Q'(s) = \sum_{y\in\Gamma_0 x_0}\|\mu_y\|\; e^{\int_{x_0}^{y}(\widetilde{F}-s)}$ converges.

If the series $Q'(\sigma)$ converges, let us prove that $\mu_{x_0}(\Lambda_c\Gamma_0) = 0$, which gives the last assertion by contraposition. For every $n \in \mathbb{N}$, define

$$A_n = \bigcap_{k\in\mathbb{N}}\;\bigcup_{y\in\Gamma_0 x_0 - B(x_0,k)}\mathscr{O}_{x_0}B(y,n)\;.$$

For every $n \in \mathbb{N}$, by Equation (153) in the proof of Proposition 11.1 (with $K = \{x_0\}$) when n is big enough, there exists $c_n > 0$ such that, for every $k \in \mathbb{N}$,

$$\mu_{x_0}(A_n) \leq \sum_{y\in\Gamma_0 x_0 - B(x_0,k)}\mu_{x_0}(\mathscr{O}_{x_0}B(y,n)) \leq c_n \sum_{y\in\Gamma_0 x_0 - B(x_0,k)}\|\mu_y\|\; e^{\int_{x_0}^{y}(\widetilde{F}-\sigma)}\;.$$

Since the sum on the right hand side tends to 0 as $k \to +\infty$, we have $\mu_{x_0}(A_n) = 0$ for every $n \in \mathbb{N}$. Since $\Lambda_c\Gamma_0 = \bigcup_{n\in\mathbb{N}} A_n$ by the definition of the conical limit set, we have $\mu_{x_0}(\Lambda_c\Gamma_0) = 0$, as required. $\Box$

Here is another consequence of Proposition 11.1, due to [Rob3, Lem. 1.2.3] when $F = 0$. For every $r > 0$ and every nonelementary discrete group Γ_0 of isometries of $\widetilde{M}$, let $\Lambda_{c,r}\Gamma_0$ be the set of elements $\xi \in \partial_\infty\widetilde{M}$ such that there exist $\rho < r$ and $(\gamma_n)_{n\in\mathbb{N}}$ in Γ_0 such that $(\gamma_n x_0)_{n\in\mathbb{N}}$ converges to ξ and $d(\gamma_n x_0, [x_0,\xi[) < \rho$ (or equivalently $\xi \in \mathscr{O}_{x_0}B(\gamma_n x_0, \rho)$). Note that the sets $\Lambda_{c,r}\Gamma_0$ are Γ_0-invariant (by the properties of asymptotic geodesic rays), are nondecreasing in r, and satisfy $\Lambda_c\Gamma_0 = \bigcup_{r>0}\Lambda_{c,r}\Gamma_0$.

Corollary 11.4 *Let Γ_0 be a discrete subgroup of $\operatorname{Isom}(\widetilde{M})$, which contains and normalises Γ, such that $\widetilde{F}$ is Γ_0-invariant. Let $\sigma \in \mathbb{R}$ and let $(\mu_x)_{x\in\widetilde{M}}$ be a Patterson density for (Γ, F) of dimension σ. Then for every $r > R_{\{x_0\}}(\sigma)$, we have*

$$\mu_{x_0}(\Lambda_c\Gamma_0 - \Lambda_{c,\,r}\Gamma_0) = 0 \,.$$

Proof. Let $\rho = R_{\{x_0\}}(\sigma)$ and $r > \rho$. First assume that the quasi-invariant measure μ_{x_0} is ergodic for the action of Γ. If $\mu_{x_0}(\Lambda_c\Gamma_0) = 0$, the result is clear. Otherwise, $\mu_{x_0}({}^c\Lambda_c\Gamma_0) = 0$ by ergodicity, and $\mu_{x_0}(\Lambda_{c,\,R}\Gamma_0) > 0$ for $R > \rho$ large enough, so that again by ergodicity $\mu_{x_0}({}^c\Lambda_{c,\,R}\Gamma_0) = 0$.

Recall without proof the following Vitali covering argument.

Lemma 11.5 *(Roblin [Rob3, Lem. 1.2.1]) For every $s > 0$ and for every discrete subset Z of $\widetilde{M}$, there exists a subset Z^* of Z such that the shadows $\mathscr{O}_{x_0}B(z,s)$ for $z \in Z^*$ are pairwise disjoint and $\bigcup_{z\in Z}\mathscr{O}_{x_0}B(z,s) \subset \bigcup_{z\in Z^*}\mathscr{O}_{x_0}B(z,5s)$.*

For every $n \in \mathbb{N}$, applying this lemma with $s = R$ and $Z_n = \Gamma_0 x_0 - B(x_0, n)$ gives a subset Z_n^* of Z_n. Since $R > \rho$, the shadows $\mathscr{O}_{x_0}B(z,\rho)$ for $z \in Z_n^*$ are pairwise disjoint. Define

$$A_n = \bigcup_{z\in Z_n^*}\mathscr{O}_{x_0}B(z,\rho) \,.$$

We have $\sigma \geq \delta_{\Gamma,\,F}$ by Corollary 3.11 (2). By Proposition 11.1 and the definition of ρ, by Equation (153), there exist $c, C > 0$ such that, for every $n \in \mathbb{N}$, we have

$$\begin{aligned}
\mu_{x_0}(A_n) &= \sum_{z\in Z_n^*}\mu_{x_0}\big(\mathscr{O}_{x_0}B(z,\rho)\big) \geq \frac{1}{C}\sum_{z\in Z_n^*}\|\mu_z\|\; e^{\int_{x_0}^{z}(\widetilde{F}-\sigma)} \\
&\geq \frac{1}{c\,C}\sum_{z\in Z_n^*}\mu_{x_0}\big(\mathscr{O}_{x_0}B(z,5R)\big) \geq \frac{1}{c\,C}\mu_{x_0}\Big(\bigcup_{z\in Z_n}\mathscr{O}_{x_0}B(z,R)\Big) \\
&\geq \frac{1}{c\,C}\mu_{x_0}(\Lambda_{c,\,R}\Gamma_0) = \frac{\|\mu_{x_0}\|}{c\,C} \,.
\end{aligned}$$

Since $r > \rho$, the set $\Lambda_{c,\,r}\Gamma_0$ contains the nonincreasing intersection of $\bigcup_{z\in Z_n}\mathscr{O}_{x_0}B(z,\rho)$ (which contains A_n). Hence $\mu_{x_0}(\Lambda_{c,\,r}\Gamma_0) \geq \frac{\|\mu_{x_0}\|}{c\,C(\sigma)} > 0$, and again by ergodicity, we have $\mu_{x_0}({}^c\Lambda_{c,\,r}\Gamma_0) = 0$.

By a Krein-Millman type of argument, since the map $(\mu_x)_{x\in\widetilde{M}} \mapsto \mu_{x_0}$ is a bijection from the set of Patterson densities for (Γ, F) of dimension σ to a convex cone of finite quasi-invariant measures on $\partial_\infty\widetilde{M}$, whose extremal points are the ergodic ones, the result follows. $\square$

11.2 Characters and critical exponents

Recall that a (real) *character* of a group G is a group morphism from G to the additive group $\mathbb{R}$.

Note that the kernel $\operatorname{Ker}\chi$ of a character χ of Γ is a nonelementary discrete group of isometries of $\widetilde{M}$, since otherwise Γ is the extension of an abelian group by a virtually nilpotent group, hence is virtually solvable, hence is elementary.

In this subsection, we fix a character χ of Γ. Define the *(twisted) Poincaré series of* (Γ, F, χ) as

$$Q_{\Gamma,\,F,\,\chi}(s) = Q_{\Gamma,\,F,\,\chi,\,x,\,y}(s) = \sum_{\gamma\in\Gamma} e^{\chi(\gamma)+\int_x^{\gamma y}(\widetilde{F}-s)} \;.$$

The *(twisted) critical exponent* of (Γ, F, χ) is the element $\delta_{\Gamma,\,F,\,\chi}$ in $[-\infty, +\infty]$ defined by

$$\delta_{\Gamma,\,F,\,\chi} = \limsup_{n\to+\infty} \frac{1}{n} \log \sum_{\gamma\in\Gamma,\; n-1<d(x,\,\gamma y)\leq n} e^{\chi(\gamma)+\int_x^{\gamma y}\widetilde{F}} \;.$$

If $\delta_{\Gamma,\,F,\,\chi} < +\infty$ (we will prove in the following Proposition 11.6 that $\delta_{\Gamma,\,F,\,\chi} > -\infty$), we say that (Γ, F, χ) is *of divergence type* if the series $Q_{\Gamma,\,F,\,\chi}(\delta_{\Gamma,\,F,\,\chi})$ diverges, and *of convergence type* otherwise.

Since $\widetilde{F}$ is Hölder-continuous, the critical exponent $\delta_{\Gamma,\,F,\,\chi}$ of (Γ, F, χ) does not depend on x, y, and we will prove below that the above upper limit is a limit. When $F = 0$, these objects were introduced in [Rob3], and we will extend in this chapter the results of this reference, which itself extends results of [BaL].

For every $s \geq 0$, for all $x, y \in \widetilde{M}$ and for all open subsets U and V of $\partial_\infty \widetilde{M}$, let

$$G_{\Gamma,\,F,\,\chi,\,x,\,y,\,U,\,V}(t) = \sum_{\gamma\in\Gamma\,:\,d(x,\gamma y)\leq t\;,\;\gamma y\in\mathscr{C}_x U\;,\;\gamma^{-1}x\in\mathscr{C}_y V} e^{\chi(\gamma)+\int_x^{\gamma y}\widetilde{F}} \;.$$

The map $s \mapsto G_{\Gamma,\,F,\,\chi,\,x,\,y,\,U,\,V}(s)$ will be called the *(twisted) bisectorial orbital counting function* of (Γ, F, U, V). When $V = \partial_\infty \widetilde{M}$, we denote it by $s \mapsto G_{\Gamma,\,F,\,\chi,\,x,\,y,\,U}(s)$ and call it the *(twisted) sectorial orbital counting function* of (Γ, F, χ, U). When $U = V = \partial_\infty \widetilde{M}$, we denote it by $s \mapsto G_{\Gamma,\,F,\,\chi,\,x,\,y}(s)$ and call it the *(twisted) orbital counting function* of (Γ, F, χ).

Let g be a periodic orbit of the geodesic flow on T^1M. Choose γ_g one of the (finitely many up to conjugation) hyperbolic elements of Γ such that if x is a point of the translation axis of γ_g, then g is obtained by first lifting to $T^1\widetilde{M}$ (by the unit tangent vectors) the geodesic segment $[x, \gamma_g x]$ oriented from x to $\gamma_g x$ and then by taking the image by $T^1\widetilde{M} \to T^1M$. Define $\chi(g) = \chi(\gamma_g)$, which does not depend on the choice of γ_g, since for all $\gamma, \gamma' \in \Gamma$, we have $\chi(\gamma) = \chi(\gamma')$ if γ and γ' are conjugated or if $\gamma'\gamma^{-1}$ has finite order. For every relatively compact open subset W of T^1M meeting the (topological) nonwandering set $\Omega\Gamma$, define the *(twisted) period counting series* of (Γ, F, χ, W) (see Subsection 4.1 when $\chi = 0$) as

$$Z_{\Gamma,\,F,\,\chi,\,W}(s) = \sum_{g\in\mathscr{P}\mathrm{er}(s),\; g\cap W\neq\emptyset} e^{\chi(g)+\int_g F} \;.$$

The *(twisted) Gurevich pressure* of (Γ, F, χ) is

$$P_{Gur}(\Gamma,\ F,\ \chi) = \limsup_{s\to+\infty} \frac{1}{s} \log Z_{\Gamma,\,F,\,\chi,\,W}(s) \;.$$

We will prove below that the twisted Gurevich pressure does not depend on W and that the above upper limit is a limit.

Remark. Assume in this remark that Γ is torsion free and that every closed differential 1-form on M is cohomologous to a (uniformally locally) Hölder-continuous one (this second

assumption is for instance satisfied if M is compact). As explained for instance in [Bab1] in a particular case, the interpretation of the characters of Γ as closed 1-forms on M allows us to reduce the above objects twisted by a character to ones associated to another potential, as we shall now see.

Given a differential 1-form ω, which is a smooth map from TM to $\mathbb{R}$ (linear on each fiber of π) that we assume to be Hölder-continuous, its restriction to T^1M, which will also be denoted by ω, is a (smooth) potential on T^1M. Note that there are potentials which do not come from differential 1-forms.

If two Hölder-continuous differential 1-forms ω and ω' are cohomologous (as differential 1-forms), and if $f : M \to \mathbb{R}$ is a smooth map such that $\omega' - \omega = df$, then their associated potentials ω and ω' are cohomologous (as potentials) via the map $G = f \circ \pi : T^1M \to \mathbb{R}$.

Let $H^1(M;\mathbb{R})$ be the space of cohomology classes of closed differential 1-forms on M. Consider the map

$$[\omega] \mapsto \Big(\gamma \mapsto \int_\gamma w = \int_x^{\gamma x} \widetilde{\omega}\Big) ,$$

where x is any point of $\widetilde{M}$ and $\widetilde{\omega}$ is the potential associated as explained above to the lift $p^*\omega$ of ω to $\widetilde{M}$ by the canonical projection $p : \widetilde{M} \to M$. By Hurewicz's theorem, this map from $H^1(M;\mathbb{R})$ to $\mathrm{Hom}(\Gamma, \mathbb{R})$ is a linear isomorphism. For every $\chi \in \mathrm{Hom}(\Gamma, \mathbb{R})$, let ω_χ be a Hölder-continuous differential 1-form on M such that $[\omega_\chi]$ maps to χ, and let $\widetilde{\omega}_\chi$ be its lift to $\widetilde{M}$.

Under the hypotheses of this remark, we then immediately have

$$Q_{\Gamma,\,F,\,\chi,\,x,\,x}(s) = Q_{\Gamma,\,F+\omega_\chi,\,x,\,x}(s) ,$$

$$\delta_{\Gamma,\,F,\,\chi} = \delta_{\Gamma,\,F+\omega_\chi} ,$$

and

$$P_{Gur}(\Gamma, F, \chi) = P_{Gur}(\Gamma, F + \omega_\chi) .$$

When $F = 0$, we recover the Poincaré series associated with a cohomology class of Γ introduced in [Bab1], and the twisted critical exponent is then called the *cohomological pressure*.

After this remark, let us give elementary properties satisfied by the twisted critical exponents.

Proposition 11.6 *Let χ be a character of Γ.*
(i) The Poincaré series of (Γ, F, χ) converges if $s > \delta_{\Gamma,\,F,\,\chi}$ and diverges if $s < \delta_{\Gamma,\,F,\,\chi}$. The Poincaré series of (Γ, F, χ) diverges at $\delta_{\Gamma,\,F,\,\chi}$ if and only if the Poincaré series of $(\Gamma, F + \kappa, \chi)$ diverges at $\delta_{\Gamma,\,F+\kappa,\,\chi}$, for every $\kappa \in \mathbb{R}$, and in particular

$$\forall\, \kappa \in \mathbb{R}, \quad \delta_{\Gamma,\,F+\kappa,\,\chi} = \delta_{\Gamma,\,F,\,\chi} + \kappa .$$

(ii) We have

$$\forall\, s \in \mathbb{R},\ Q_{\Gamma,\,F\circ\iota,\,-\chi,\,x,\,y}(s) = Q_{\Gamma,\,F,\,\chi,\,y,\,x}(s) \quad \text{and} \quad \delta_{\Gamma,\,F\circ\iota,\,-\chi} = \delta_{\Gamma,\,F,\,\chi} .$$

(iii) If Γ' is a nonelementary subgroup of Γ, denoting by $F' : \Gamma'\backslash T^1M \to \mathbb{R}$ the map induced by $\widetilde{F}$, and by χ' the restriction to Γ' of χ, we have

$$\delta_{\Gamma',\,F',\,\chi'} \leq \delta_{\Gamma,\,F,\,\chi} .$$

(iv) We have

$$\forall\, s\in\mathbb{R},\ Q_{\Gamma,\,F,\,\chi,\,x,\,y}(s)+Q_{\Gamma,\,F\circ\iota,\,\chi,\,y,\,x}(s)\geq Q_{\Gamma,\,F,\,x,\,y}(s) \tag{155}$$

and hence

$$\max\{\,\delta_{\Gamma,\,F,\,\chi},\ \delta_{\Gamma,\,F\circ\iota,\,\chi}\}\geq\delta_{\Gamma,\,F}\ .$$

(v) If there exists $c\geq 0$ such that $|\chi(\gamma)|\leq c\,d(x_0,\gamma x_0)$ for every $\gamma\in\Gamma$, then

$$\delta_{\Gamma,\,F}-c\ \leq\ \delta_{\Gamma,\,F,\,\chi}\ \leq\ \delta_{\Gamma,\,F}+c\,.$$

(vi) We have $\delta_{\Gamma,\,F,\,\chi}>-\infty$.

(vii) The map $(F,\chi)\mapsto\delta_{\Gamma,\,F,\,\chi}$ is convex, that is, if χ^* is another character of Γ, if $\widetilde{F^*}:T^1\widetilde{M}\to\mathbb{R}$ is another Hölder-continuous Γ-invariant map, inducing $F^*:\Gamma\backslash T^1M\to\mathbb{R}$, if $\delta_{\Gamma,\,F,\,\chi}$ and $\delta_{\Gamma,\,F^*,\,\chi^*}$ are finite, then for every $t\in[0,1]$, we have

$$\delta_{\Gamma,\,tF+(1-t)F^*,\,t\chi+(1-t)\chi^*}\leq t\,\delta_{\Gamma,\,F,\,\chi}+(1-t)\,\delta_{\Gamma,\,F^*,\,\chi^*}\ .$$

(viii) We have

$$\delta_{\Gamma,\,F,\,\chi}=\limsup_{n\to+\infty}\ \frac{1}{n}\log\sum_{\gamma\in\Gamma,\ d(x,\,\gamma y)\leq n}e^{\chi(\gamma)+\int_x^{\gamma y}\widetilde{F}}\ .$$

(ix) If $\widetilde{F^*}:T^1\widetilde{M}\to\mathbb{R}$ is another Hölder-continuous Γ-invariant map, which is cohomologous to $\widetilde{F}$, and if $F^*:T^1M\to\mathbb{R}$ is the induced map, then

$$\delta_{\Gamma,\,F,\,\chi}=\delta_{\Gamma,\,F^*,\,\chi}\ .$$

Before giving a proof, here are a few immediate consequences. It follows from Assertion (iv) that if F is reversible, then

$$\delta_{\Gamma,\,F,\,-\chi}=\delta_{\Gamma,\,F,\,\chi}\geq\delta_{\Gamma,\,F}\ . \tag{156}$$

We claim that the assumption of Assertion (v) is satisfied for instance if Γ is convex-cocompact. More generally, assume that Γ is finitely generated (which is the case if it is convex-cocompact), and let S be a finite generating set of Γ. For every $\gamma\in\Gamma$, let $\|\gamma\|_S$ be the smallest length of a word in $S\cup S^{-1}$ representing γ. For every character χ of Γ, we have

$$|\chi(\gamma)|\leq\|\gamma\|_S\ \max_{s\in S}|\chi(s)|\ .$$

Assume also that the map $\gamma\mapsto\gamma x_0$ from Γ (endowed with the word distance $(\gamma,\gamma')\mapsto\|\gamma^{-1}\gamma'\|_S$) to $\widetilde{M}$ is quasi-isometric, that is, assume that there exists $c>0$ such that for every $\gamma\in\Gamma$, we have

$$d(x_0,\gamma x_0)\geq c\|\gamma\|_S\ .$$

This property does not depend on x_0 and is satisfied for instance if Γ is convex-cocompact. Then the assumption of Assertion (v) is indeed satisfied.

If Γ satisfies the hypothesis of Assertion (v) and if $\delta_{\Gamma,\,F}$ is finite (by Lemma 3.3 (iv), this is for instance the case if $\widetilde{F}$ is bounded on $\pi^{-1}(\mathscr{C}\Lambda\Gamma)$, which itself is the case if Γ is convex-cocompact), then $\delta_{\Gamma,\,F,\,\chi}$ is finite.

In particular, if Γ is convex-cocompact, then $\delta_{\Gamma,\,F,\,\chi}$ is finite for every character χ of Γ.

Proof. The proof is an immediate adaptation of the proof of Lemma 3.3. For instance, to prove Equation (155), we use

$$\sum_{\gamma\in\Gamma} e^{\chi(\gamma)+\int_x^{\gamma y}(\widetilde{F}-s)} + \sum_{\gamma\in\Gamma} e^{\chi(\gamma^{-1})+\int_y^{\gamma^{-1}x}(\widetilde{F}\circ\iota-s)} = \sum_{\gamma\in\Gamma} \big(e^{\chi(\gamma)}+e^{-\chi(\gamma)}\big)\, e^{\int_x^{\gamma y}(\widetilde{F}-s)}$$
$$\geq \sum_{\gamma\in\Gamma} e^{\int_x^{\gamma y}(\widetilde{F}-s)} . \quad \Box$$

The following results concerning the twisted critical exponent and twisted Gurevich pressure have proofs similar to the ones when $\chi=0$ seen in Chapter 4.

Theorem 11.7 *(1) If $\delta_{\Gamma,F,\chi}>0$, we have $G_{\Gamma,F,\chi,x,y,U,V}(t)=O(e^{t\,\delta_{\Gamma,F,\chi}})$ as t goes to $+\infty$.*

(2) We have

$$\delta_{\Gamma,F,\chi} = \lim_{n\to+\infty}\frac{1}{n}\log \sum_{\gamma\in\Gamma,\ d(x,\gamma y)\leq n} e^{\chi(\gamma)+\int_x^{\gamma y}\widetilde{F}} = \lim_{s\to+\infty}\frac{1}{s}\log\ G_{\Gamma,F,\chi,x,y}(s) .$$

(3) If U is an open subset of $\partial_\infty\widetilde{M}$ meeting $\Lambda\Gamma$, then

$$\lim_{t\to+\infty}\frac{1}{t}\log G_{\Gamma,F,\chi,x,y,U}(t)=\delta_{\Gamma,F,\chi} .$$

(4) If U and V are any two open subsets of $\partial_\infty\widetilde{M}$ meeting the limit set $\Lambda\Gamma$, then

$$\lim_{t\to+\infty}\frac{1}{t}\log G_{\Gamma,F,\chi,x,y,U,V}(t)=\delta_{\Gamma,F,\chi} .$$

(5) Let W be a relatively compact open subset of T^1M meeting the nonwandering set $\Omega\Gamma$. Then

$$P_{Gur}(\Gamma,F,\chi)=\lim_{s\to+\infty}\frac{1}{s}\log Z_{\Gamma,F,\chi,W}(s)=\delta_{\Gamma,F,\chi} ,$$

Proof. (1) See the proof of Corollary 4.1 (using the twisted Patterson measure constructed in the coming Subsection 11.3 and the extension of Mohsen's shadow lemma in Proposition 11.1).

(2) See the proof of Theorem 4.2 (and also the sub-additivity argument of Dal'bo-Peigné-Sambusetti in its following remark).

(3) See the proof of Corollary 4.3 (1), where Equation (58) should be replaced by

$$a_{t,U',z}=e^{-\chi(\gamma)}\ a_{t,\gamma U',\gamma z} ,$$

and the constant c_2' in Equation (60) should be replaced by $c_2'+\max_{1\leq i\leq k}\ |\chi(\gamma_i)|$.

(4) See the proof of Corollary 4.3 (2), with adjustements similar to those above, in particular since now

$$b_{t,U',V',z,w}=e^{\chi(\alpha)}\ b_{t,U',\alpha V',z,\alpha w} .$$

(5) See the proof of Theorem 4.4, using the fact that for all $\gamma,\gamma'\in\Gamma$, we have $\chi(\gamma)=\chi(\gamma')$ if $\gamma'\gamma^{-1}$ has finite order. $\Box$

11.3 Characters and Patterson densities

Let $\chi : \Gamma \rightarrow \mathbb{R}$ be a character of Γ.

For every $\sigma \in \mathbb{R}$, a *twisted Patterson density* of dimension σ for (Γ, F, χ) is a family of finite nonzero (positive Borel) measures $(\mu_x)_{x\in\widetilde{M}}$ on $\partial_\infty\widetilde{M}$ such that, for every $\gamma \in \Gamma$, for all $x, y \in \widetilde{M}$, for every $\xi \in \partial_\infty\widetilde{M}$, we have

$$\gamma_*\mu_x = e^{-\chi(\gamma)}\,\mu_{\gamma x}\,, \tag{157}$$

$$\frac{d\mu_x}{d\mu_y}(\xi) = e^{-C_{F-\sigma,\,\xi}(x,y)}\,. \tag{158}$$

This definition is due to Babillot-Ledrappier [BaL] and Roblin [Rob3] when $F = 0$.

Note that if Γ' is a nonelementary subgroup of $\operatorname{Ker}\chi$ (for instance $\Gamma' = \operatorname{Ker}\chi$), denoting by $F' : \Gamma'\backslash T^1M \rightarrow \mathbb{R}$ the map induced by $\widetilde{F}$, then a twisted Patterson density of dimension σ for (Γ, F, χ) is a (standard) Patterson density of dimension σ for (Γ', F').

Similarly as in Subsection 3.6, for every twisted Patterson density $(\mu_x)_{x\in\widetilde{M}}$ of dimension σ for (Γ, F, χ), we have :

- $(\mu_x)_{x\in\widetilde{M}}$ is a twisted Patterson density of dimension $\sigma + s$ for $(\Gamma, F + s, \chi)$, for every $s \in \mathbb{R}$.
- the support of μ_x, which is independant of $x \in \widetilde{M}$, contains $\Lambda\Gamma$.

The twisted Patterson densities satisfy the following elementary properties, due to [Rob3] when $F = 0$.

Proposition 11.8 *Let χ be a character of Γ.*

(1) If $\delta_{\Gamma,F,\chi} < +\infty$, then there exists a twisted Patterson density of dimension $\delta_{\Gamma,F,\chi}$ for (Γ, F, χ), whose support is exactly $\Lambda\Gamma$.

(2) For every $\sigma \in \mathbb{R}$, if there exists a twisted Patterson density of dimension σ for (Γ, F, χ), then $\sigma \geq \delta_{\Gamma,F,\chi}$.

(3) If Γ' is a nonelementary normal subgroup of Γ, with $F' : \Gamma'\backslash T^1M \rightarrow \mathbb{R}$ the map induced by $\widetilde{F}$, if $(\mu'_x)_{x\in\widetilde{M}}$ is a Patterson density of dimension σ for (Γ', F'), which is ergodic for the action of Γ' and quasi-invariant for the action of Γ, then there exists a character χ of Γ, whose kernel contains Γ', such that $(\mu'_x)_{x\in\widetilde{M}}$ is a twisted Patterson density of dimension σ for (Γ, F, χ).

(4) For every $\sigma \in \mathbb{R}$ such that there exists a twisted Patterson density of dimension σ for (Γ, F, χ), and for all $x, y \in \widetilde{M}$, there exists $c > 0$ such that for every $n \in \mathbb{N}$, we have

$$\sum_{\gamma\in\Gamma\,:\,n-1<d(x,\gamma y)\leq n} e^{\chi(\gamma)+\int_x^{\gamma y}(\widetilde{F}-\sigma)} \leq c\,.$$

Proof. (1) As in [Rob3] when $F = 0$ and in [Moh] when $\chi = 0$, the construction is similar to Patterson's one (see the proof of Proposition 3.9). For every $z \in \widetilde{M}$, let $\mathscr{D}_z$ be the unit Dirac mass at z. Let $h : [0, +\infty[\, \rightarrow\,]0, +\infty[$ be a nondecreasing map such that

- for every $\epsilon > 0$, there exists $r_\epsilon \geq 0$ such that $h(t + r) \leq e^{\epsilon t}h(r)$ for all $t \geq 0$ and $r \geq r_\epsilon$;
- for all $x, y \in \widetilde{M}$, if $\overline{Q}_{x,y}(s) = \sum_{\gamma\in\Gamma} e^{\chi(\gamma)+\int_x^{\gamma y}(\widetilde{F}-s)}\,h(d(x, y))$, then $\overline{Q}_{x,y}(s)$ diverges if and only if $s \leq \delta_{\Gamma,F,\chi}$.

Such a map is easy to construct, by taking $\log h$ to be piecewise affine on $[0,+\infty[$, with positive slope tending to 0 near $+\infty$ (when the series $Q_{\Gamma,\,F,\,\chi,\,x,\,y}(s)$ diverges at $s=\delta_{\Gamma,\,F,\,\chi}$, we may take $h=1$ constant).

For $s>\delta=\delta_{\Gamma,\,F,\,\chi}$, define the measure

$$\mu_{x,\,s}=\frac{1}{\overline{Q}_{x_0,\,x_0}(s)}\sum_{\gamma\in\Gamma}\;e^{\chi(\gamma)+\int_x^{\gamma x_0}(\widetilde{F}-s)}\;h(d(x,x_0))\;\mathscr{D}_{\gamma x_0}$$

on $\widetilde{M}$. By the weak star compactness of the space of probability measures on the compact space $\widetilde{M}\cup\partial_\infty\widetilde{M}$, there exists a sequence $(s_k)_{k\in\mathbb{N}}$ in $]\delta,+\infty[$ converging to δ such that the sequence of measures $(\mu_{x_0,\,s_k})_{k\in\mathbb{N}}$ weak star converges to a measure μ_{x_0} on $\widetilde{M}\cup\partial_\infty\widetilde{M}$, with support contained in $\Lambda\Gamma$, hence equal to $\Lambda\Gamma$ by minimality. For every $x\in\widetilde{M}$, defining

$$d\mu_x(\xi)=e^{-C_{F-\delta,\,\xi}(x,\,x_0)}\;d\mu_{x_0}(\xi)\;,$$

it is easy to see that $(\mu_x)_{x\in\widetilde{M}}=\lim_{k\to+\infty}(\mu_{x,\,s_k})_{x\in\widetilde{M}}$ is a twisted Patterson density of dimension δ for (Γ,F,χ), with support $\Lambda\Gamma$.

(2) Let $(\mu_x)_{x\in\widetilde{M}}$ be a twisted Patterson density of dimension σ for (Γ,F,χ). Note that $\|\mu_{\gamma x_0}\|=e^{\chi(\gamma)}\|\mu_{x_0}\|$ by Equation (157). The result hence follows from the first claim of Corollary 11.3 applied by taking (Γ_0,Γ) therein to be $(\Gamma,\operatorname{Ker}\chi)$.

(3) The proof is the same as the one when $F=0$ in [Rob3, Lem. 2.1.1 (c)].

(4) The proof is similar to the one when $\chi=0$ in Corollary 3.11 (1), using Proposition 11.1 instead of Mohsen's shadow lemma 3.10. $\Box$

Remark. Let $\widetilde{F}^*:T^1\widetilde{M}\to\mathbb{R}$ be a Hölder-continuous Γ-invariant map, which is cohomologous to $\widetilde{F}$ via the Γ-invariant map $\widetilde{G}:T^1\widetilde{M}\to\mathbb{R}$. Let $(\mu_x)_{x\in\widetilde{M}}$ be a twisted Patterson density of dimension $\sigma\in\mathbb{R}$ for (Γ,F,χ). Then the family of measures $(\mu^*_x)_{x\in\widetilde{M}}$ defined by, for every $\xi\in\partial_\infty\widetilde{M}$,

$$d\mu^*_x(\xi)=e^{\widetilde{G}(v_{x\xi})}\;d\mu_x(\xi)\;,$$

where $v_{x\xi}$ is the tangent vector at the origin of the geodesic ray from a point $x\in\widetilde{M}$ to ξ, is a twisted Patterson density of dimension also σ for (Γ,F^*,χ).

11.4 Galois covers and critical exponents

Let Γ' be a nonelementary subgroup of Γ, and let $F':\Gamma'\backslash T^1M\to\mathbb{R}$ be the map induced by $\widetilde{F}$.

The aim of this subsection is to compare the critical exponents of (Γ,F) and (Γ',F'). We saw in Lemma 3.3 that

$$\delta_{\Gamma',\,F'}\le\delta_{\Gamma,\,F}\;.\tag{159}$$

The case of equality in Equation (159) is related to the amenability properties of the left quotient $\Gamma'\backslash\Gamma$, in the following sense.

We endow the left quotient $\Gamma'\backslash\Gamma$ with the discrete topology, and with the action (on the right) of Γ by translations on the right. We denote by $R_\alpha:[\gamma]\mapsto[\gamma\alpha]$ the action of $\alpha\in\Gamma$ on $\Gamma'\backslash\Gamma$. A *right-invariant mean* $\mathfrak{m}$ on the left quotient $\Gamma'\backslash\Gamma$ is a linear map $\mathfrak{m}:\ell^\infty(\Gamma'\backslash\Gamma)\to\mathbb{R}$ such that $\mathfrak{m}(1)=1$, $\mathfrak{m}(f)\ge 0$ if $f\ge 0$ and $\mathfrak{m}(f\circ R_\alpha)=\mathfrak{m}(f)$ for

every $\alpha \in \Gamma$. The left quotient $\Gamma'\backslash\Gamma$ is *amenable* if it has a right-invariant mean. When Γ' is normal in Γ, there are many equivalent definitions, see for instance [Gre, Pate]. For instance, if Γ' is normal in Γ and if the group $\Gamma'\backslash\Gamma$ is virtually solvable, then the left quotient $\Gamma'\backslash\Gamma$ is amenable.

When $F = 0$, the following result is due to [Rob3, Théo. 2.2.2]. We suspect that the normality assumption on Γ' and the reversibility one on F might not be necessary.

Theorem 11.9 *If F is reversible, if Γ' is a nonelementary normal subgroup of Γ and if $\Gamma'\backslash\Gamma$ is amenable, then $\delta_{\Gamma',\,F'} = \delta_{\Gamma,\,F}$.*

Proof. Let $\mathfrak{m} : \ell^\infty(\Gamma'\backslash\Gamma)$ be a right-invariant mean on the left quotient $\Gamma'\backslash\Gamma$. We may assume that $\delta' = \delta_{\Gamma',\,F'} < +\infty$, otherwise the result follows from Equation (159).

Let $(\mu_x)_{x\in\widetilde{M}}$ be a Patterson density of dimension δ' for (Γ', F'), which exists as seen in Proposition 3.9. For every $\alpha \in \Gamma$, we claim that there exists a map $f_\alpha \in \ell^\infty(\Gamma'\backslash\Gamma)$ such that, for every $[\gamma] \in \Gamma'\backslash\Gamma$,

$$f_\alpha([\gamma]) = \log \|\mu_{\gamma\alpha x_0}\| - \log \|\mu_{\gamma x_0}\| \;.$$

Since $(\mu_x)_{x\in\widetilde{M}}$ is Γ'-equivariant, for every $\gamma \in \Gamma$, the value of $f_\alpha([\gamma])$ indeed depends only on the class of γ in $\Gamma'\backslash\Gamma$. By Lemma 3.4 (1), there exists $c_1 > 0$ such that

$$\begin{aligned}
|\, f_\alpha([\gamma])\,| &\leq \max_{\xi\in\partial_\infty\widetilde{M}} \big|\, C_{F-\delta',\,\xi}(\gamma\alpha x_0, \gamma x_0)\,\big| \\
&\leq c_1\, e^{d(\alpha x_0,\, x_0)} \;+\; d(\alpha x_0, x_0) \max_{\pi^{-1}(B(\alpha x_0,\, d(\alpha x_0,\, x_0)))} |\widetilde{F} - \delta'| \;.
\end{aligned}$$

Hence f_α indeed belongs to $\ell^\infty(\Gamma'\backslash\Gamma)$.

For all $\alpha, \beta, \gamma \in \Gamma$, we have

$$f_{\alpha\beta}([\gamma]) = \log \|\mu_{\gamma\alpha\beta x_0}\| - \log \|\mu_{\gamma\alpha x_0}\| + \log \|\mu_{\gamma\alpha x_0}\| - \log \|\mu_{\gamma x_0}\|$$

Therefore $f_{\alpha\beta} = f_\beta \circ R_\alpha + f_\alpha$, and the map $\chi : \Gamma \to \mathbb{R}$ defined by $\chi(\alpha) = \mathfrak{m}(f_\alpha)$ is a character of Γ.

For every $x \in \widetilde{M}$ and every continuous map $\varphi : \partial_\infty\widetilde{M} \to \mathbb{R}$, let $f_{x,\,\varphi} \in \ell^\infty(\Gamma'\backslash\Gamma)$ be the map defined by

$$f_{x,\,\varphi}([\gamma]) = \frac{1}{\|\mu_{\gamma x_0}\|} \int \varphi \; d\,(\gamma^{-1})_*\mu_{\gamma x} \;.$$

This map is well defined, and bounded by an argument similar to the above one. By Riesz's theorem, there exists a (positive Borel) measure ν_x on $\partial_\infty\widetilde{M}$ such that

$$\int \varphi \; d\nu_x = \mathfrak{m}(f_{x,\,\varphi})$$

for every continuous map $\varphi : \partial_\infty\widetilde{M} \to \mathbb{R}$. For all $x \in \widetilde{M}$, $\gamma \in \Gamma$, $\gamma' \in \Gamma'$ and $\varphi : \partial_\infty\widetilde{M} \to \mathbb{R}$ continuous, using the fact that Γ' is normal in Γ, we have

$$\|\mu_{\gamma\gamma' x_0}\| = \|\mu_{(\gamma\gamma'\gamma^{-1})\gamma x_0}\| = \|(\gamma\gamma'\gamma^{-1})_*\mu_{\gamma x_0}\| = \|\mu_{\gamma x_0}\| \quad \text{and} \quad f_{\gamma' x,\,\varphi} = f_{x,\,\varphi\circ\gamma'} \circ R_{\gamma'} \;.$$

Using this and Lemma 11.2, it is then easy to check that $(\nu_x)_{x\in\widetilde{M}}$ is also a Patterson density of dimension δ' for (Γ', F').

Since the proof of Jensen's inequality does not require the measure to be σ-additive, by the convexity of the exponential map, for every $f \in \ell^\infty(\Gamma'\backslash\Gamma)$, we have $e^{\mathfrak{m}(f)} \leq \mathfrak{m}(e^f)$. In particular, for every $\alpha \in \Gamma$, we have

$$e^{\chi(\alpha)} = e^{\mathfrak{m}(f_\alpha)} \leq \mathfrak{m}(e^{f_\alpha}) = \mathfrak{m}(f_{\alpha x_0,\,1}) = \|\nu_{\alpha x_0}\| \;.$$

By the first assertion of Corollary 11.3 applied by taking $(\Gamma_0, \Gamma, (\mu_x)_{\in\widetilde{M}})$ therein equal to $(\Gamma, \Gamma', (\nu_x)_{\in\widetilde{M}})$, we have $\delta' \geq \delta_{\Gamma,\,F,\,\chi}$. Since F is reversible, we have $\delta_{\Gamma,\,F,\,\chi} \geq \delta_{\Gamma,\,F}$ by Equation (156). Hence $\delta' \geq \delta_{\Gamma,\,F}$, and the result follows from Equation (159). □

A nonelementary normal subgroup Γ' of Γ cannot be too small. For instance, we have seen that the inclusion $\Lambda\Gamma' \subset \Lambda\Gamma$ is an equality. Similarly, the default of the inequality (159) to be an equality cannot be too large, even if $\Gamma'\backslash\Gamma$ is nonamenable. If $\widetilde{M}$ is a real hyperbolic space $\mathbb{H}^n_\mathbb{R}$ with $n \geq 3$ or a complex hyperbolic space $\mathbb{H}^n_\mathbb{C}$ with $n \geq 2$, and if Γ is a lattice in $\mathrm{Isom}(\widetilde{M})$, then Shalom [Sha, Theo. 1.4] has proved some results of this type. When $F = 0$, the following result is due to [Rob3, Théo. 2.2.1].

Theorem 11.10 *(1) If Γ' is any nonelementary normal subgroup of Γ, then*

$$\delta_{\Gamma',\,F'} \geq \frac{1}{2}\,\delta_{\Gamma,\,F+F\circ\iota}\;.$$

(2) If F is reversible, if $\delta_{\Gamma,\,2F} < 2\;\delta_{\Gamma,\,F}$ and if $(\Gamma, 2F)$ is of divergence type, then $\delta_{\Gamma',\,F'} > \frac{1}{2}\,\delta_{\Gamma,\,2F}$.

Note that the condition that $\delta_{\Gamma,\,F+F\circ\iota} < 2\;\delta_{\Gamma,\,F}$ is satisfied for instance if the upper bound $\sup_{\pi^{-1}(\mathscr{C}\Lambda\Gamma)} |F|$ is small enough, by Lemma 3.3 (iv).

Proof. Let $\delta = \delta_{\Gamma,\,F+F\circ\iota}$ and $\delta' = \delta_{\Gamma',\,F'}$. If F is reversible, then $F + F\circ\iota$ is cohomologous to $2F$, hence $\delta_{\Gamma,\,F+F\circ\iota} = \delta_{\Gamma,\,2F}$ and $(\Gamma, 2F)$ is of divergence type if and only if $(\Gamma, F + F\circ\iota)$ is of divergence type (see the remark at the end of Subsection 3.2).

We may assume that $\delta' < +\infty$. Let $(\mu'_x)_{x\in\widetilde{M}}$ be a Patterson density of dimension δ' for (Γ', F'), such that the dynamical system $(\partial_\infty\widetilde{M}, \Gamma', \mu'_{x_0})$ is ergodic, which exists as seen in Subsection 3.6. Let $R' = R_{\{x_0\}}(\delta')$ and $C' = C_{\{x_0\}}(\delta')$ be given by Proposition 11.1 applied by taking $(\Gamma_0, \Gamma, (\mu_x)_{x\in\widetilde{M}})$ therein equal to $(\Gamma, \Gamma', (\mu'_x)_{x\in\widetilde{M}})$.

(1) For every $y \in \Gamma x_0$, Proposition 11.1 gives that

$$\|\mu'_y\| \geq \mu'_y\big(\mathscr{O}_y B(x_0, R')\big) \geq \frac{\|\mu'_{x_0}\|}{C'}\; e^{\int_y^{x_0}(\widetilde{F}-\delta')}\;. \tag{160}$$

Hence for every $s > \delta'$, we have

$$\sum_{\gamma\in\Gamma} e^{\int_{x_0}^{\gamma x_0}(\widetilde{F}+\widetilde{F}\circ\iota-\delta'-s)} \leq \frac{C'}{\|\mu'_{x_0}\|} \sum_{\gamma\in\Gamma} \|\mu'_{\gamma x_0}\|\; e^{\int_{x_0}^{\gamma x_0}(\widetilde{F}-s)}\;.$$

By the first claim of Corollary 11.3, the sum on the right hand side converges. This proves that $\delta \leq 2\,\delta'$.

(2) Assume for a contradiction that $\delta' = \frac{\delta}{2}$. Since $(\Gamma, F + F\circ\iota)$ is of divergence type, let $(\mu_x)_{x\in\widetilde{M}}$ be a Patterson density of dimension δ for $(\Gamma, F + F\circ\iota)$, which gives full measure

to $\Lambda_c\Gamma$ (see Corollary 5.12). By Corollary 11.4 applied with $\Gamma_0 = \Gamma$, let $r \geq R'$ be large enough so that μ_{x_0} gives full measure to $\Lambda_{c,\,r}\Gamma$.

Let us first prove that μ_{x_0} is absolutely continuous with respect to μ'_{x_0}. Let B be any Borel subset of $\partial_\infty \widetilde{M}$, and let V be any open neighbourhood of B. Let

$$Z = \{z \in \Gamma x_0 \;:\; \mathscr{O}_{x_0} B(z,r) \subset V\}\;,$$

and let Z^* be the subset of Z constructed in Lemma 11.5 with $s = r$. Using respectively

- the fact that $\Lambda_{c,\,r}\Gamma$ has full measure with respect to μ_{x_0},
- the fact that $B \cap \Lambda_{c,\,r}\Gamma$ is contained in $\bigcup_{z\in Z} \mathscr{O}_{x_0} B(z,r)$ and Lemma 11.5,
- Equation (153), which provides a constant $c > 0$ depending only on $R = 5r$ and $K = \{x_0\}$,
- the Γ-equivariance of $(\mu_x)_{x\in\widetilde{M}}$, which implies that $\|\mu_{\gamma x_0}\| = \|\mu_{x_0}\|$ for every $\gamma \in \Gamma$,
- Equation (160),
- the equality $\delta = 2\delta'$,
- Proposition 11.1 applied by taking $(\Gamma_0, \Gamma, (\mu_x)_{\in\widetilde{M}})$ therein equal to $(\Gamma, \Gamma', (\mu'_x)_{\in\widetilde{M}})$, since $r \geq R'$, and setting $c' = \frac{c\,(C')^2\,\|\mu_{x_0}\|}{\|\mu'_{x_0}\|}$,
- the properties of Z^* seen in Lemma 11.5 and the definition of Z,

we have

$$\begin{aligned}
\mu_{x_0}(B) = \mu_{x_0}(B \cap \Lambda_{c,\,r}\Gamma) &\leq \sum_{z\in Z^*} \mu_{x_0}\big(\mathscr{O}_{x_0} B(z, 5r)\big)\\
&\leq \sum_{z\in Z^*} c\,\|\mu_z\|\, e^{\int_{x_0}^{z}(\widetilde{F}+\widetilde{F}\circ\iota-\delta)} = c\,\|\mu_{x_0}\| \sum_{z\in Z^*} e^{\int_{x_0}^{z}(\widetilde{F}+\widetilde{F}\circ\iota-\delta)}\\
&\leq c\,\|\mu_{x_0}\| \sum_{z\in Z^*} \frac{C'\|\mu'_z\|}{\|\mu'_{x_0}\|}\, e^{-\int_{z}^{x_0}(\widetilde{F}-\delta')}\, e^{\int_{x_0}^{z}(\widetilde{F}+\widetilde{F}\circ\iota-\delta)}\\
&= \frac{c\,C'\,\|\mu_{x_0}\|}{\|\mu'_{x_0}\|} \sum_{z\in Z^*} \|\mu'_z\|\, e^{\int_{x_0}^{z}(\widetilde{F}-\delta')}\\
&\leq c' \sum_{z\in Z^*} \mu'_{x_0}\big(\mathscr{O}_{x_0} B(z,r)\big) \leq c'\,\mu'_{x_0}(V)\;.
\end{aligned}$$

Since V is an arbitrary neighbourhood of B, we hence have $\mu_{x_0} \leq c'\,\mu'_{x_0}$, as required.

Now, the Radon-Nikodym derivative $\frac{d\,\mu_{x_0}}{d\,\mu'_{x_0}}$ is quasi-invariant under Γ' and the measure μ_{x_0} is not the zero measure. By ergodicity of $(\partial_\infty\widetilde{M}, \Gamma', \mu'_{x_0})$, the map $\frac{d\,\mu_{x_0}}{d\,\mu'_{x_0}}$ is hence a positive constant μ'_{x_0}-almost everywhere. Therefore μ'_{x_0} is quasi-invariant under Γ, as is μ_{x_0}. By Proposition 11.8 (3), there exists a character χ of Γ such that $(\mu'_x)_{x\in\widetilde{M}}$ is a twisted Patterson density of dimension δ' for (Γ, F, χ). Hence $\delta' \geq \delta_{\Gamma,\,F,\,\chi}$ by Proposition 11.8 (2). Since F is reversible, we have $\delta_{\Gamma,\,F,\,\chi} \geq \delta_{\Gamma,\,F}$ by Equation (156). Since $\delta_{\Gamma,\,F} > \frac{\delta}{2}$ by assumption, we have $\delta' > \frac{\delta}{2}$, a contradiction. $\square$

11.5 Characters and the Hopf-Tsuji-Sullivan-Roblin theorem

Let χ be a character of Γ and $\sigma \in \mathbb{R}$. Let $(\mu^\iota_x)_{x\in\widetilde{M}}$ and $(\mu_x)_{x\in\widetilde{M}}$ be twisted Patterson densities of the same dimension σ for $(\Gamma, F\circ\iota, -\chi)$ and (Γ, F, χ) respectively.

Using the Hopf parametrisation $v \mapsto (v_-, v_+, t)$ with respect to the base point x_0 and Equation (25), we define the *(twisted) Gibbs measure on $T^1\widetilde{M}$ associated with the pair of twisted Patterson densities* $\big((\mu^\iota_x)_{x\in\widetilde{M}}, (\mu_x)_{x\in\widetilde{M}}\big)$ as the measure $\widetilde{m}$ on $T^1\widetilde{M}$ given by

$$\begin{aligned} d\,\widetilde{m}(v) &= \frac{d\mu^\iota_x(v_-)\, d\mu_x(v_+)\, dt}{D_{F-\sigma,\,x}(v_-, v_+)^2} \\ &= e^{C_{F\circ\iota-\sigma,\,v_-}(x,\,\pi(v)) + C_{F-\sigma,\,v_+}(x,\,\pi(v))}\; d\mu^\iota_x(v_-)\, d\mu_x(v_+)\, dt\;. \end{aligned}$$

The measure $\widetilde{m}$ on $T^1\widetilde{M}$ is independent of $x \in \widetilde{M}$ by the equations (158) and (21), hence is invariant under the action of Γ by the equations (26) and (157) (the cancellation $e^{-\chi(\gamma)}e^{\chi(\gamma)} = 1$ is the reason why it is important that the Patterson density $(\mu^\iota_x)_{x\in\widetilde{M}}$ is twisted by the opposite character $-\chi$). It is invariant under the geodesic flow. Hence it defines a measure m on T^1M which is invariant under the quotient geodesic flow, called the *(twisted) Gibbs measure on T^1M associated with the pair of twisted Patterson densities* $\big((\mu^\iota_x)_{x\in\widetilde{M}}, (\mu_x)_{x\in\widetilde{M}}\big)$.

The measure $\iota_*\widetilde{m}$ is the twisted Gibbs measure on $T^1\widetilde{M}$ associated with the switched pair of twisted Patterson densities $\big((\mu_x)_{x\in\widetilde{M}}, (\mu^\iota_x)_{x\in\widetilde{M}}\big)$ (and similarly on T^1M).

The following results have proofs similar to those of Theorem 5.3 and Theorem 5.4.

Theorem 11.11 *The following assertions are equivalent.*

(1) The twisted Poincaré series of (Γ, F, χ) at the point σ converges: $Q_{\Gamma,\,F,\,\chi}(\sigma) < +\infty$.

(2) There exists $x \in \widetilde{M}$ such that $\mu_x(\Lambda_c\Gamma) = 0$.

(2') For every $x \in \widetilde{M}$, we have $\mu_x(\Lambda_c\Gamma) = 0$.

(3) There exists $x \in \widetilde{M}$ such that $\mu^\iota_x(\Lambda_c\Gamma) = 0$.

(3') For every $x \in \widetilde{M}$, we have $\mu^\iota_x(\Lambda_c\Gamma) = 0$.

(4) There exists $x \in \widetilde{M}$ such that the dynamical system $(\partial^2_\infty\widetilde{M}, \Gamma, (\mu^\iota_x \otimes \mu_x)_{|\partial^2_\infty\widetilde{M}})$ is non-ergodic and completely dissipative.

(5) The dynamical system $(T^1M, (\phi_t)_{t\in\mathbb{R}}, m)$ is non-ergodic and completely dissipative.

Theorem 11.12 *The following assertions are equivalent.*

(i) The twisted Poincaré series of (Γ, F, χ) at the point σ diverges: $Q_{\Gamma,\,F,\,\chi}(\sigma) = +\infty$.

(ii) There exists $x \in \widetilde{M}$ such that $\mu_x({}^c\Lambda_c\Gamma) = 0$.

(ii') For every $x \in \widetilde{M}$, we have $\mu_x({}^c\Lambda_c\Gamma) = 0$.

(iii) There exists $x \in \widetilde{M}$ such that $\mu^\iota_x({}^c\Lambda_c\Gamma) = 0$.

(iii') For every $x \in \widetilde{M}$, we have $\mu^\iota_x({}^c\Lambda_c\Gamma) = 0$.

(iv) There exists $x \in \widetilde{M}$ such that the dynamical system $(\partial^2_\infty\widetilde{M}, \Gamma, (\mu^\iota_x \otimes \mu_x)_{|\partial^2_\infty\widetilde{M}})$ is ergodic and completely conservative.

(v) The dynamical system $(T^1M, (\phi_t)_{t\in\mathbb{R}}, m)$ is ergodic and completely conservative.

Proof. Let us give a few words on the proofs of these two results, besides referring to Subsection 5.2. We use Proposition 11.1 with (Γ_0, Γ) therein replaced by $(\Gamma, \operatorname{Ker}\chi)$, coupled with Equation (153), instead of Mohsen's shadow lemma 3.10. In particular, using Proposition 11.8 (4) instead of Lemma 3.11 (1), Lemma 5.6 now becomes the following statement:

Let $x \in \widetilde{M}$, if $Q_{\Gamma,\,F,\,\chi}(\sigma) = +\infty$ and if R is large enough, with $K = K(x, R)$, there exists $c'_8 > 0$ such that for every sufficiently large $T > 0$

$$\int_0^T\int_0^T \Big(\sum_{\alpha,\,\beta\in\Gamma} \widetilde{m}(K\cap\phi_{-t}\alpha K\cap\phi_{-s-t}\beta K)\Big)\,ds\,dt \le c'_8\Big(\sum_{\gamma\in\Gamma,\,d(x,\,\gamma x)\le T} e^{\chi(\gamma)+\int_x^{\gamma x}(\widetilde{F}-\sigma)}\Big)^2 .$$

Similarly, Lemma 5.7 becomes:

Let $x \in \widetilde{M}$, if $Q_{\Gamma,\,F,\,\chi}(\sigma) = +\infty$ and if R is large enough, with $K = K(x, R)$, there exists $c'_9 > 0$ such that for every sufficiently large $T > 0$

$$\int_0^T \Big(\sum_{\gamma\in\Gamma} \widetilde{m}(K\cap\phi_{-t}\gamma K)\Big)\,dt \ge c'_9 \sum_{\gamma\in\Gamma,\,d(x,\,\gamma x)\le T} e^{\chi(\gamma)+\int_x^{\gamma x}(\widetilde{F}-\sigma)} .$$

We do not know if Assertion (g) of Proposition 5.5 is still valid (its proof no longer is). But the combination of (f) and (g), that is the fact that if $Q_{\Gamma,\,F,\,\chi}(\sigma) = +\infty$ then $\mu_x({}^c\Lambda_c\Gamma) = 0$ and $\mu^\iota_x({}^c\Lambda_c\Gamma) = 0$, is still valid, by using the fact that Assertion (f) is valid for every twisted Patterson density, and by considering the family $(\mathbb{1}_{{}^c\Lambda_c\Gamma}\mu_x)_{x\in\widetilde{M}}$ which, when $\mu_x({}^c\Lambda_c\Gamma) > 0$, is also a twisted Patterson density for (Γ, F, χ).

Now the deduction of Theorem 5.3 and Theorem 5.4 from Proposition 5.5, which only uses this combination of (f) and (g), remains valid. $\Box$

As when $\chi = 0$ in Subsection 5.3, the following results are extensions of respectively Corollary 5.10, Corollary 5.12 and Corollary 5.14.

Corollary 11.13 *(1) Assume that $\delta_{\Gamma,\,F,\,\chi} < +\infty$ and that there exists a twisted Patterson density $(\mu_x)_{x\in X}$ for (Γ, F, χ) of dimension $\sigma \ge \delta_{\Gamma,\,F,\,\chi}$ such that $\mu_{x_0}(\Lambda_c\Gamma) > 0$, then $\sigma = \delta_{\Gamma,\,F,\,\chi}$ and (Γ, F, χ) is of divergence type.*

(2) Assume that $\delta_{\Gamma,\,F,\,\chi} < +\infty$ and that (Γ, F, χ) is of divergence type.

a) There exists a twisted Patterson density $(\mu_x)_{x\in X}$ for (Γ, F, χ) of dimension $\delta_{\Gamma,\,F,\,\chi}$, which is unique up to a scalar multiple. The measures μ_x are ergodic with respect to Γ, and give full measure to $\Lambda_c\Gamma$. In particular their support is $\Lambda\Gamma$.

b) If $\mathscr{D}_z$ is the unit Dirac mass at a point $z \in \widetilde{M}$, then for every $x \in \widetilde{M}$, we have

$$\frac{\mu_x}{\|\mu_x\|} = \lim_{s\to\delta_{\Gamma,\,F,\,\chi}{}^+} \frac{1}{Q_{\Gamma,\,F,\,\chi,\,x,\,x}(s)} \sum_{\gamma\in\Gamma} e^{\chi(\gamma)+\int_x^{\gamma x}(\widetilde{F}-s)}\,\mathscr{D}_{\gamma x} . \tag{161}$$

c) We have $\mu_x(\Lambda\Gamma - \Lambda_{\rm Myr}\Gamma) = 0$.

(3) Let $\sigma \in \mathbb{R}$. If there exists a twisted Gibbs measure of dimension σ for (Γ, F, χ) which is finite on T^1M, then $\sigma = \delta_{\Gamma,\,F,\,\chi}$, the triple (Γ, F, χ) is of divergence type, there exists a unique (up to a scalar multiple) twisted Gibbs measure with potential F on T^1M, its support is $\Omega\Gamma$, and the geodesic flow $(\phi_t)_{t\in\mathbb{R}}$ on T^1M is ergodic with respect to m_F. $\Box$

11.6 The Fatou-Roblin radial convergence theorem

We state in this subsection the main tool for the classification of Patterson densities on nilpotent covers of M, to be given in Subsection 11.7. It is (an immediate extension of) one of the main results of [Rob3], that we will call the *Fatou-Roblin radial convergence* theorem. Recall that a sequence of points $(y_i)_{i\in\mathbb{N}}$ in $\widetilde{M}$ converges *radially* to a point $\xi \in \partial_\infty \widetilde{M}$ if it converges to ξ while staying at bounded distance from a geodesic ray.

Recall that Fatou's (ratio) radial convergence theorem says that given g and h two positive harmonic functions on the open unit disc $\mathbb{D}^2$ in $\mathbb{R}^2$, the ratio $g(z)/h(z)$ has a limit when $z \in \mathbb{D}^2$ radially converges to a point ζ on the unit circle $\mathbb{S}^1$ almost everywhere. Consider a nonelementary Fuchsian group Γ' that is, when $\mathbb{D}^2$ is endowed with Poincaré's metric (and hence is a real hyperbolic plane), a non virtually cyclic discrete subgroup of isometries of $\mathbb{D}^2$. If $(\mu_x)_{x\in\mathbb{D}^2}$ is a Patterson density for Γ' (without potential), then Sullivan has proved that the map $x \mapsto \|\mu_x\|$ is harmonic. Hence the ratio of the total masses of two such Patterson densities has radial limits almost everywhere.

The next result extends this, and we refer to [Rob3] for more motivation.

Theorem 11.14 (Fatou-Roblin radial convergence theorem) *Let Γ_0 be a discrete subgroup of $\operatorname{Isom}(\widetilde{M})$, which contains and normalises Γ, such that $\widetilde{F}$ is Γ_0-invariant. Let $\sigma \in \mathbb{R}$ and let $(\mu_x)_{x\in\widetilde{M}}$ and $(\nu_x)_{x\in\widetilde{M}}$ be two Patterson densities of dimension σ for (Γ, F). Then μ_{x_0} is absolutely continuous with respect to ν_{x_0} on the conical limit set of Γ_0, and for ν_{x_0}-almost every $\xi \in \Lambda_c\Gamma_0$, if $(y_i)_{i\in\mathbb{N}}$ is a sequence in $\Gamma_0 x_0$ converging radially to ξ, then*

$$\lim_{i\to+\infty} \frac{\|\mu_{y_i}\|}{\|\nu_{y_i}\|} = \frac{d\,\mu_{x_0}}{d\,\nu_{x_0}}(\xi)\ .$$

Proof. The proof of [Rob3, Théo. 1.2.2] extends almost immediately, except the first of its five steps, that needs some adaptation.

Let $R = R_{\{x_0\}}(\sigma)$ and $C = C_{\{x_0\}}(\sigma)$ be given by Proposition 11.1 with $K = \{x_0\}$. We endow any infinite subset of $\Gamma_0 x_0$ with its *Fréchet filter* of the complementary sets of its finite subsets. Let us fix $r \in [R, +\infty[$. For every $\xi \in \Lambda_{c,\,r}\Gamma_0$, let

$$D_r^+(\xi) = \lim_{\rho\to r^-}\ \limsup_{y\in\Gamma_0 x_0,\ d(y,\,[x_0,\xi[)\le\rho} \frac{\|\mu_y\|}{\|\nu_y\|}\ .$$

This limit does exist, since the function of ρ is nondecreasing, and $D_r^+ : \Lambda_{c,\,r}\Gamma_0 \to \mathbb{R}\cup\{+\infty\}$ is a Borel Γ-invariant map, by the properties of asymptotic geodesic rays and the Γ-equivariance property of $(\mu_x)_{x\in\widetilde{M}}$ and $(\nu_x)_{x\in\widetilde{M}}$.

We claim that if $D_r^+(\xi) \ge 1$ for ν_{x_0}-almost every $\xi \in \Lambda_{c,\,r}\Gamma_0$, then there exists $c' > 0$ such that $\mu_{x_0} \ge c'\nu_{x_0}$ on $\Lambda_{c,\,r}\Gamma_0$.

Given this claim, the remainder of the proof of [Rob3, Théo. 1.2.2] yields without modification the proof of Theorem 11.14.

To prove this claim, for every Borel subset B of $\partial_\infty\widetilde{M}$, and every open neighbourhood V of B, let

$$Z = \{z \in \Gamma_0 x_0\ :\ \mathscr{O}_{x_0}B(z,r) \subset V\ \text{ and }\ \|\mu_z\| \ge \frac{1}{2}\|\nu_z\|\}\ ,$$

and let Z^* be the subset of Z given by Lemma 11.5 with $s = r$. The assumption of the above claim implies that, up to a set of zero ν_{x_0}-measure, the set $B\cap\Lambda_{c,\,r}\Gamma_0$ is contained in

$\bigcup_{z\in Z} \mathscr{O}_{x_0} B(z,r)$. Hence, by the properties of Z^*, for the constant $c>0$ given by Equation (153), which depends only on $5r$ and $K=\{x_0\}$, and by Proposition 11.1, we have

$$\begin{aligned}\nu_{x_0}(B\cap\Lambda_{c,\,r}\Gamma_0) &\leq \sum_{z\in Z^*}\nu_{x_0}\big(\mathscr{O}_{x_0}B(z,5r)\big)\leq c\sum_{z\in Z^*}\|\nu_z\|\;e^{\int_{x_0}^{z}(\widetilde{F}-\sigma)}\\ &\leq 2c\sum_{z\in Z^*}\|\mu_z\|\;e^{\int_{x_0}^{z}(\widetilde{F}-\sigma)}\leq 2c\,C\sum_{z\in Z^*}\mu_{x_0}\big(\mathscr{O}_{x_0}B(z,r)\big)\leq 2c\,C\;\mu_{x_0}(V)\;.\end{aligned}$$

Since V is an arbitrary open neighbourhood of B, we have $\nu_{x_0}(B\cap\Lambda_{c,\,r}\Gamma_0)\leq 2c\,C\;\mu_{x_0}(B)$, which proves the claim. □

The next result, especially interesting when Γ_0 is convex-cocompact, gives a criterion for a Patterson density of (Γ,F) to be determined by its total masses at given points.

Corollary 11.15 *Let Γ_0 be a discrete subgroup of $\operatorname{Isom}(\widetilde{M})$, which contains and normalises Γ, such that $\widetilde{F}$ is Γ_0-invariant.*

(1) Let $\sigma,\sigma'\in\mathbb{R}$. Let $(\mu_x)_{x\in\widetilde{M}}$ and $(\nu_x)_{x\in\widetilde{M}}$ be two Patterson densities for (Γ,F) of dimension σ and σ' respectively, giving full measure to $\Lambda_c\Gamma_0$. If for every $y\in\Gamma_0x_0$ we have $\|\mu_y\|=\|\nu_y\|$, then $(\mu_x)_{x\in\widetilde{M}}=(\nu_x)_{x\in\widetilde{M}}$ (and in particular $\sigma=\sigma'$).

(2) Assume that $\delta_{\Gamma_0,\,F}<+\infty$ and that (Γ_0,F) is of divergence type. For every character χ of Γ_0, if $\delta_{\Gamma_0,\,F}=\delta_{\Gamma_0,\,F,\,\chi}$, then $\chi=0$.

(3) If χ is a character of Γ, two twisted Patterson densities for (Γ,F,χ), giving full measure to $\Lambda_c\Gamma$, are proportional by a positive constant, and in particular have the same dimension.

Proof. The proof of the first two points is the same as when $F=0$ (see [Rob3, Coro 1.2.5] and [Rob3, Lem. 2.1.2]). The third point is immediate from the first one applied with (Γ_0,Γ) equal to $(\Gamma,\operatorname{Ker}\chi)$. □

11.7 Classification of ergodic Patterson densities on nilpotent covers

Let Γ' be a normal subgroup of Γ, and let $F':\Gamma'\backslash T^1M\to\mathbb{R}$ be the map induced by $\widetilde{F}$.

We start by recalling the definition of a nilpotent group. Let G be a group. If H and H' are subgroups of G, let $[H,H']$ be the subgroup of G generated by the elements $[h,h']=hh'h^{-1}h'^{-1}$ with $h\in H$ and $h'\in H'$. The *lower central series* of a group G is the sequence of normal subgroups $(G_n)_{n\in\mathbb{N}}$ defined by induction by

$$G_0=G\quad\text{and}\quad G_{n+1}=[G,G_n]\ \text{ for }\ n\in\mathbb{N}\;.$$

The group G is *nilpotent* if there exists $n\in\mathbb{N}$ such that $G_n=\{e\}$. It is well known that a nilpotent group is in particular amenable.

Theorem 11.16 *Let Γ' be a normal subgroup of Γ such that Γ/Γ' is nilpotent. Assume that $\delta_{\Gamma,\,F}<+\infty$. For every $\sigma\in[\delta_{\Gamma,\,F},+\infty[$, the set of Patterson densities of (Γ',F') of dimension σ, giving full measure to $\Lambda_c\Gamma$ and ergodic with respect to the action of Γ', is equal to the set of twisted Patterson densities of (Γ,F,χ) of dimension σ for the characters χ of Γ vanishing on Γ' such that $\delta_{\Gamma,\,F,\,\chi}=\sigma$, giving full measure to $\Lambda_c\Gamma$.*

Proof. As usual with nilpotent groups, the proof proceeds by induction on the nilpotency order. By taking the preimages of the terms of the lower central series of Γ/Γ', there exist $n \in \mathbb{N}$ and subgroups Γ_k of Γ for $k = 0, \dots, n$ such that $\Gamma_0 = \Gamma, \Gamma_n = \Gamma'$, Γ_{k+1} is a normal subgroup of Γ_k and $\Gamma_{k+1}/\Gamma' = [\Gamma/\Gamma', \Gamma_k/\Gamma']$ for $k = 0, \dots, n-1$. We denote by $F_k : \Gamma_k \backslash T^1\widetilde{M} \to \mathbb{R}$ the map induced by $\widetilde{F}$. Let us fix $\sigma \geq \delta_{\Gamma,\,F}$.

If $n = 0$, then $\Gamma = \Gamma'$ and the only character of Γ vanishing on Γ' is $\chi = 0$. The result is then immediate, as we have seen in Corollary 5.10 that if there exists a Patterson density $(\mu_x)_{x\in X}$ for (Γ, F) of dimension $\sigma \geq \delta_{\Gamma,\,F}$ such that $\mu_x(\Lambda_c\Gamma) > 0$, then $\sigma = \delta_{\Gamma,\,F}$ and (Γ, F) is of divergence type, hence the measures μ_x are ergodic for the action of Γ by Theorem 5.4. Therefore we assume that $n \geq 1$.

Step 1. There exists $c > 0$ such that for every $k \geq 1$, for every Patterson density $(\mu_x)_{x\in\widetilde{M}}$ of dimension σ for (Γ_k, F_k), giving full measure to $\Lambda_c\Gamma$, and for all $\alpha \in \Gamma_{k-1}$, we have

$$\alpha_*\mu_{\alpha x_0} \geq c\; e^{\int_{x_0}^{\alpha x_0}(\widetilde{F}\circ\iota - \sigma)}\; \mu_{x_0}\;.$$

Proof. Let $c = \frac{1}{C_{\{x_0\}}(\sigma)} > 0$, with the notation of Proposition 11.1. For all $\gamma \in \Gamma$ and $\alpha \in \Gamma_{k-1}$, since $[\alpha, \gamma] \in [\Gamma_{k-1}, \Gamma]$ belongs to Γ_k, we have, by this proposition,

$$\begin{aligned}\|\alpha_*\mu_{\alpha\gamma x_0}\| = \|\mu_{\alpha\gamma x_0}\| = \|\mu_{\gamma\alpha x_0}\| &\geq \frac{1}{C_{\{x_0\}}(\sigma)}\;\|\mu_{\gamma x_0}\|\; e^{\int_{\gamma\alpha x_0}^{\gamma x_0}(\widetilde{F}-\sigma)}\\ &= c\;\|\mu_{\gamma x_0}\|\; e^{\int_{x_0}^{\alpha x_0}(\widetilde{F}\circ\iota-\sigma)}\;.\end{aligned}$$

By Lemma 11.2 since Γ_{k-1} normalises Γ_k, and by the Fatou-Roblin radial convergence theorem 11.14 applied by replacing (Γ_0, Γ) therein by (Γ, Γ_{k-1}), this proves the first step. $\Box$

Step 2. Let $\mu = (\mu_x)_{x\in\widetilde{M}}$ be a Patterson density of dimension σ for (Γ', F'), giving full measure to $\Lambda_c\Gamma$, and ergodic for the action of Γ'. Then there exists a character χ of Γ, vanishing on Γ', such that μ is a twisted Patterson density of dimension σ for (Γ, F, χ).

Proof. Let us first prove that, for $k \in \{2, \dots, n\}$, if μ is a Patterson density for (Γ_k, F_k), then μ is a Patterson density for (Γ_{k-1}, F_{k-1}).

Since μ_{x_0} is ergodic under Γ' hence under Γ_k, and since μ_{x_0} is absolutely continuous with respect to $\alpha_*\mu_{\alpha x_0}$ for every $\alpha \in \Gamma_{k-1}$ by Step 1, there exists a character χ_{k-1} of Γ_{k-1}, trivial on Γ_k, such that $\mu_{x_0} = e^{-\chi_{k-1}(\alpha)}\;\alpha_*\mu_{\alpha x_0}$ for every $\alpha \in \Gamma_{k-1}$. For every $\alpha \in \Gamma_{k-1}$, we have

$$\chi_{k-1}(\alpha) = \log\|\mu_{\alpha x}\| - \log\|\mu_x\|$$

for every $x \in \widetilde{M}$ (by the definition of χ_{k-1} for $x = x_0$ and by Lemma 11.2 and Equation (35) to extend to any x).

Since Γ/Γ' is amenable, let $\mathfrak{m}$ be a right-invariant mean on $\Gamma'\backslash\Gamma = \Gamma/\Gamma'$. As seen in the proof of Theorem 11.9, for every $\alpha \in \Gamma$, the map f_α which associates to $[\gamma] \in \Gamma/\Gamma'$ the real number $\log\|\mu_{\gamma\alpha x_0}\| - \log\|\mu_{\gamma x_0}\|$ is well defined and bounded, and the map $\chi : \Gamma \to \mathbb{R}$ defined by $\chi(\alpha) = \mathfrak{m}(f_\alpha)$ is a character of Γ. This character hence coincides with χ_{k-1} on Γ_{k-1}, by applying the above centered formula to $x = \gamma x_0$ for every $\gamma \in \Gamma$ and since, as seen in the proof of Step 1, $\|\mu_{\gamma\alpha x_0}\| = \|\mu_{\alpha\gamma x_0}\|$ for every $\gamma \in \Gamma$. Since $\Gamma_{k-1}/\Gamma' \subset [\Gamma/\Gamma', \Gamma_{k-2}/\Gamma']$, the additive character χ vanishes on Γ_{k-1}, hence $\chi_{k-1} = 0$. This proves the preliminary claim.

Now, by induction, we hence have that μ is a Patterson density for (Γ_1, F_1). Applying again Step 1 gives that μ is quasi-invariant under Γ. Applying Proposition 11.8 (3) (using for this the ergodicity of μ_{x_0} under Γ' hence under Γ_1), the second step follows. $\Box$

Step 3. If χ is a character of Γ vanishing on Γ', if $(\mu_x)_{x\in\widetilde{M}}$ is a twisted Patterson density of dimension σ for (Γ, F, χ), giving full measure to $\Lambda_c\Gamma$, then $\sigma = \delta_{\Gamma,\,F,\,\chi}$ and μ_{x_0} is ergodic for the action of Γ'.

Proof. Let A be a Borel subset of $\Lambda_c\Gamma$ invariant under Γ', with nonzero μ_{x_0}-measure, and let $\mathbb{1}_A$ be its characteristic function. Then $(\mathbb{1}_A\,\mu_x)_{x\in\widetilde{M}}$ is a Patterson density of dimension σ for (Γ', F'), giving full measure to $\Lambda_c\Gamma$. The preliminary claim in the proof of Step 2 shows that $(\mathbb{1}_A\,\mu_x)_{x\in\widetilde{M}}$ is also a Patterson density of dimension σ for (Γ_1, F_1). By Step 1, $\mathbb{1}_A\,\mu_x$ is quasi-invariant under Γ, hence A is invariant up to a set of measure 0 under Γ. In particular, $(\mathbb{1}_A\,\mu_x)_{x\in\widetilde{M}}$ is also a twisted Patterson density of dimension σ for (Γ, F, χ), giving full measure to $\Lambda_c\Gamma$. By their uniqueness property seen in Corollary 11.13 (1) and (2), we have $\sigma = \delta_{\Gamma,\,F,\,\chi}$ and there exists $c > 0$ such that $\mathbb{1}_A\,\mu_{x_0} = c\,\mu_{x_0}$. In particular $\mu_{x_0}({}^c\!A) = 0$, which proves the ergodicity of μ_{x_0} under Γ'. $\Box$

Step 2 and Step 3 combine together to prove Theorem 11.16. $\Box$

The following result is then immediate. The assertions (2) and (3) of Theorem 1.10 in the introduction follow from it and from Corollary 11.13.

Corollary 11.17 *Let Γ' be a normal subgroup of Γ such that Γ/Γ' is nilpotent.*

(1) Assume that $\delta_{\Gamma,\,F} < +\infty$ and that (Γ, F) is of divergence type. Then the unique (up to a scalar multiple) Patterson density $(\mu_{\Gamma,\,F,\,x})_{x\in\widetilde{M}}$ of dimension $\delta_{\Gamma,\,F}$ for (Γ, F) is also the unique (up to a scalar multiple) Patterson density of dimension $\delta_{\Gamma,\,F}$ for (Γ', F') giving full measure to $\Lambda_c\Gamma$.

(2) If Γ is convex-cocompact, then the set of ergodic Patterson densities for (Γ', F') is the set of twisted Patterson densities of (Γ, F, χ) of dimension $\delta_{\Gamma,\,F,\,\chi}$ for the characters χ of Γ vanishing on Γ'.

Proof. By Corollary 5.12, there exists a unique (up to a scalar multiple) Patterson density $\mu = (\mu_{\Gamma,\,F,\,x})_{x\in\widetilde{M}}$ of dimension $\sigma = \delta_{\Gamma,\,F}$ for (Γ, F), and it gives full measure to $\Lambda_c\Gamma$. By restriction, μ is also a Patterson density of dimension σ for (Γ', F') giving full measure to $\Lambda_c\Gamma$.

Let $\nu = (\nu_x)_{x\in\widetilde{M}}$ be another Patterson density of dimension σ for (Γ', F') giving full measure to $\Lambda_c\Gamma$. Assume furthermore that ν is ergodic with respect to the action of Γ'. By Theorem 11.16, let χ be a character of Γ such that ν is a twisted Patterson density for (Γ, F, χ) of dimension $\delta_{\Gamma,\,F,\,\chi} = \sigma$. By Corollary 11.15 (2), we have $\chi = 0$. Hence ν is a scalar multiple of μ. By Krein-Milman's theorem, the ergodicity assumption on ν may be dropped. This proves Assertion (1).

(2) Since $\Lambda_c\Gamma = \Lambda\Gamma$ when Γ is convex-cocompact, Assertion (2) follows immediately from Theorem 11.16. $\Box$

We refer to [Rob3] for possible improvements of the second assertion when Γ is only assumed to be geometrically finite (with $\delta_{\Gamma,\,F} < +\infty$).

It follows from the first assertion that not only the dynamical system $(\partial_\infty\widetilde{M}, \Gamma, \mu_{\Gamma,\,F,\,x})$ is ergodic, but that the dynamical system $(\partial_\infty\widetilde{M}, \Gamma', \mu_{\Gamma,\,F,\,x})$ is also ergodic. This extends

results of Babillot-Ledrappier [BaL] when $F = 0$ and Γ'/Γ is isomorphic to $\mathbb{Z}^d$ for $d \in \mathbb{N}$, of Hamenstädt [Ham4] when M is compact, and of Roblin [Rob3] when $F = 0$.

Remark. Let Γ' be a subgroup of Γ. Assume that $\delta_{\Gamma,\,F} < +\infty$ and that (Γ, F) is of divergence type. Let $\widetilde{m}_F$ be the Gibbs measure of $(\Gamma,\ F)$ on $T^1\widetilde{M}$ (which is unique up to a scalar multiple). Let $\phi = (\phi_t)_{t\in\mathbb{R}}$ be the geodesic flow on the (orbifold) cover $M' = \Gamma'\backslash\widetilde{M}$ of M defined by the group Γ'. Let m'_F be the measure induced on $T^1M' = \Gamma'\backslash T^1\widetilde{M}$ by $\widetilde{m}_F$. Note that if Γ' has infinite index in Γ, the Gibbs measure m'_F is infinite.

Even if Γ' is normal and $\Gamma'\backslash\Gamma$ is nilpotent, the dynamical system

$$(T^1M', \phi, m'_F)$$

is not necessarily ergodic: the ergodicity of $(\partial_\infty\widetilde{M}, \Gamma', \mu_{\Gamma,\,F,\,x})$ does not imply the ergodicity of (T^1M', ϕ, m'_F). See [Ree] for examples.

A necessary and sufficient condition on the left quotient $\Gamma'\backslash\Gamma$ for (T^1M', ϕ, m'_F) to be ergodic is still unknown.

List of Symbols

$W^{ss}(v)$ strong stable leaf of $v \in T^1\widetilde{M}$ or $v \in T^1M$. 17

$\widetilde{\mathscr{W}}^{ss}$ strong stable foliation of $T^1\widetilde{M}$. 17

$\mathscr{W}^{ss}$ strong stable foliation of T^1M. 17

$W^{s}(v)$ stable leaf of $v \in T^1\widetilde{M}$ or $v \in T^1M$. 17

$\widetilde{\mathscr{W}}^{s}$ stable foliation of $T^1\widetilde{M}$. 17

$\mathscr{W}^{s}$ stable foliation of T^1M. 17

$W^{su}(v)$ strong unstable leaf of $v \in T^1\widetilde{M}$ or $v \in T^1M$. 17

$\widetilde{\mathscr{W}}^{su}$ strong unstable foliation of $T^1\widetilde{M}$. 17

$\mathscr{W}^{su}$ strong unstable foliation of T^1M. 17

$W^{u}(v)$ unstable leaf of $v \in T^1\widetilde{M}$ or $v \in T^1M$. 17

$\widetilde{\mathscr{W}}^{u}$ unstable foliation of $T^1\widetilde{M}$. 17

$\mathscr{W}^{u}$ unstable foliation of T^1M. 17

$Z_{\Gamma,\,F,\,W}$ period counting function of (Γ, F, W). 54

$Z_{\Gamma,\,F,\,\chi,\,W}$ (twisted) period counting function of (Γ, F, χ, W). 172

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